---

1. [10 Marks]Answer the following questions.

   (a) [2] Show that $[\hat{X}, \hat{P}] = i\hbar$, where $\hat{X}$ and $\hat{P}$ are position and momentum operators.

   (b) [2] For a particle in a box, the wave function is given by

   $$\Psi(x) = A\sqrt{x\,(L-x)}.$$

   Find A and sketch the probability density function for position measurement.

   (c) [2] What is the probability that the particle in question 1(b) will be found in the interval $[0, \frac{L}{4}]$?

   (d) [2] Show that the eigenvalues of a unitary matrix are complex numbers with unit magnitude.

   (e) [2] Let $\hat{p}$ be the momentum operator. Show that $\exp[i\hat{p}a/\hbar]f(x) = f(x+a)$ where a is a real constant. [Assume that f is a smooth function of a real variable.]

2. [10 Marks] The wave function of a free particle at some instant is given by

   $$\Psi(x) = B\exp\left[i\frac{p_0 x}{\hbar}\right]\exp\left[-\frac{|x|}{2d}\right]$$

   where, B, p_0, and d are positive real constants.

   (a) [2] Find B by normalizing Ψ.

   (b) [3] Find $\langle\hat{x}\rangle, \langle\hat{x}^2\rangle$.

   (c) [4] Find $\langle\hat{p}\rangle, \langle\hat{p}^2\rangle$. [Hint: Calculate $\langle\hat{p}^2\rangle$ by evaluating $\langle\hat{p}\Psi, \hat{p}\Psi\rangle$]

   (d) [1] Verify uncertainty principle.

3. [6 Marks] A particle is in the ground state of an infinite potential well of width L. Now the well is suddenly expanded **symmetrically** to the width of $2L$, leaving the wavefunction undisturbed. Show that the probability of finding the particle in the ground state of the new well is $(8/3\pi)^2$.

4. [4 Marks] Consider a n dimensional complex inner product space V_n with elements given by a $n \times 1$ column matrix. The inner product of two elements is defined as

   $$\langle u, v\rangle = u^\dagger v$$

   where $u^\dagger$ is the complex conjugate of the transpose of u. Let $B = \{e_1, e_2, \ldots e_n\}$ be an orthonormal basis. A family of projection operators is defined as

   $$\mathbf{P}_j u = \langle e_j, u\rangle\, e_j \qquad j = 1, 2, \ldots n.$$

   (a) [1] Show that the projection operators are hermitian.

   (b) [1] Find the matrix of $\mathbf{P}_j$ wrt the basis B.

   (c) [1] Show that $\mathbf{P}_i\mathbf{P}_j = \delta_{ij}\mathbf{P}_i$.

   (d) [1] Show that $\sum_{i=1}^{n}\mathbf{P}_i = \mathbf{I}$, where $\mathbf{I}$ is an identity operator.