1. Consider a particle trapped in a cubical box given by potential

$$V(x,y,z) = \begin{cases} 0 & 0 \le x,y,z \le L \\ \infty & \text{otherwise.} \end{cases}$$

   (a) Let $n(E)$ be the number of energy eigenstates with energy less than E. Find $n(E)$.

   (b) Find the density of states, which is defined as

$$g(E) = \frac{1}{L^3}\frac{dn}{dE}(E).$$

   (c) Sketch $g(E)$.

2. An anisotropic harmonic oscillator has the potential energy function

$$V(x,y,z) = \frac{1}{2}m\omega^2\left(x^2+y^2\right) + \frac{1}{2}m\omega_z^2 z^2.$$

   (Assume that ω_z/ω is large and irrational.)

   (a) Write down first few eigen-energies and their degeneracies.

   (b) This problem is separable in cartesian coordinates. Write down the eigenfunctions corresponding to the first few eigenstates.

   (c) Do the operators $\mathbf{L}_x$, $\mathbf{L}_y$ and $\mathbf{L}_z$ commute with the hamiltonian? And does $\mathbf{L}^2$?

   (d) Write down the ground state. Is this eigenfunction $\mathbf{L}_z$? If so, what is the eigenvalue?

   (e) The degeneracy of the first excited state (energy: $\frac{1}{2}\hbar\omega_z + 2\hbar\omega$) is two. When separated in cartesian coordinates, the *un-normalized* eigenfunctions are

$$\begin{aligned} \phi_{100}(x,y,z) &= x\exp\left[-\frac{\alpha^2x^2}{2}\right]\exp\left[-\frac{\alpha^2y^2}{2}\right]\exp\left[-\frac{\alpha_z^2z^2}{2}\right] \\ \phi_{010}(x,y,z) &= y\exp\left[-\frac{\alpha^2x^2}{2}\right]\exp\left[-\frac{\alpha^2y^2}{2}\right]\exp\left[-\frac{\alpha_z^2z^2}{2}\right] \end{aligned}$$

   where $\alpha = \sqrt{m\omega/\hbar}$ and $\alpha_z = \sqrt{m\omega_z/\hbar}$. Show that these functions are not eigenfunctions of $\mathbf{L}_z$? Can you construct linear combinations of ϕ_{100} and ϕ_{010}, which are eigenfunctions of $\mathbf{L}_z$? (Hint: Write these functions in spherical polar coordinates and remember $e^{im\phi}$ are eigenfunctions of $\mathbf{L}_z$.)

3. Let $x_1 = x$, $x_2 = y$, and $x_3 = z$. Simillary, for any vector quantity $\mathbf{A}$, let $A_1 = A_x$, $A_2 = A_y$ and $A_3 = A_z$.

   (a) Prove that $\mathbf{L}$ is a hermitian operator.

   (b) Prove $[L_i, x_j] = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}x_k$. Here ϵ_{ijk} is called Levi-Civita antisymmetric symbol, given by

$$\epsilon_{ijk} = \begin{cases} 1 & \text{if } ijk = 123, 231, 312 \\ -1 & \text{if } ijk = 132, 321, 213 \\ 0 & \text{otherwise.} \end{cases}$$

   (c) Prove $[L_i, p_j] = i\hbar\sum_{k=1}^{3}\epsilon_{ijk}p_k$.

4. For Legendre polynomials, Prove:

   (a) Orthogonality:

$$\int_{-1}^{1} P_m(x)P_n(x) = \frac{2}{2n+1}\delta_{m,n}$$

   (b) Recursion Relations:

$$\begin{aligned}(n+1)P_{n+1}(x) &= (2l+1)xP_n(x) - nP_{n-1}(x) \\ (1-x^2)\frac{dP_n}{dx} &= -nxP_n(x) + nP_{n-1}(x)\end{aligned}$$

5. Let $l > m > 0$.

   (a) Show that $P_l^m(-x) = (-1)^{l-m}P_l^m(x)$.

   (b) If the spherical coordinates of a vector $\mathbf{r}$ are (r, θ, ϕ), what are the sperical coordinates of $-\mathbf{r}$?

   (c) Show that $Y_{l,m}$ has a parity $(-1)^l$ under $\mathbf{r} \to -\mathbf{r}$ transformation.

6. Show that $\mathbf{L}_- = \hbar e^{-i\phi}\left(-\frac{\partial}{\partial\theta} + i\cot\theta\frac{\partial}{\partial\phi}\right)$. Also show that $\mathbf{L}_-Y_{l,m} = \hbar\sqrt{l\,(l+1) - m\,(m-1)}Y_{l,m-1}$. Use recurrence relation 6.97d given on page 282 of Bransden.

7. Just as $Y_{l,m}$ is a an eigenfunctions of $\mathbf{L}^2$ and $\mathbf{L}_z$, let $Z_{l,n}$ be an eigenfunction of $\mathbf{L}^2$ and $\mathbf{L}_x$ that is

$$\begin{aligned}\mathbf{L}^2 Z_{l,n} &= l(l+1)\hbar^2 Z_{l,n} \\ \mathbf{L}_x Z_{l,n} &= n\hbar Z_{l,n}.\end{aligned}$$

   We already know the eigenvalues of $\mathbf{L}^2$, that is $l = 0, 1, 2, \ldots$.

   (a) What are allowed values for n for given l?

   (b) Show that $\langle Z_{l,n}, Y_{l',m}\rangle = 0$ if $l \neq l'$.

   (c) Show that $Z_{l,n}$ can be expressed in terms of $Y_{l,m}$, that is

$$Z_{l,n} = \sum_{m=-l}^{l} C_{n,m}Y_{l,m}.$$

   (d) Find $Z_{1,n}$ in terms of $Y_{1,m}$ explicitly.

8. Let the state of a particle constrained to move on a sphere, be $Y_{1,0}$. What are the possible results of measurement of L_x? What is the probability associated with each outcome?

9. The state of a particle constrained to move on a sphere is

$$\Psi(\theta, \phi, t = 0) = \frac{1}{\sqrt{4\pi}}\left(e^{i\phi}\sin\theta + \cos\theta\right)$$

   (a) What are the probabilities for the various results of the measurement of $\mathbf{L}_z$ at $t = 0$? What about at $t > 0$?

   (b) What is the expectation value of $\mathbf{L}_z$ at $t = 0$?