1. Prove the following identities:

   (a) $[A, B + C] = [A, B] + [A, C]$.

   (b) $[A, BC] = [A, B]\,C + B\,[A, C]$.

   (c) $[A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0$.

   (d) $\left[f\left(\hat{X}\right), \hat{P}\right] = i\hbar\frac{df}{dx}$, where f is operator function of $\hat{X}$.

2. For any operator A, show that $\left(A + A^\dagger\right)$, $i\left(A - A^\dagger\right)$ and $AA^\dagger$ are hermitian operators.

3. Prove:
$$e^A B e^{-A} = B + [A, B] + \frac{1}{2!}[A, [A, B]] + \frac{1}{3!}[A, [A, [A, B]]] + \cdots$$
and if A and B commute with their commutator then
$$e^A e^B = e^{A+B+\frac{1}{2}[A,B]}.$$

4. Let $\mathcal{H}$ be a Hilbert space. Prove Schwarz inequality, that is, for any two vectors, $f, g \in \mathcal{H}$, show that
$$|\langle f, g\rangle|^2 \le \langle f, f\rangle \langle g, g\rangle\,.$$
[Hint: Consider $\langle f + \lambda g, f + \lambda g\rangle \ge 0$. Now find λ such that the lhs is minimum.]

5. Let a time dependent observable be represented by a hermitian operator$\hat{\Omega}(t)$. If the system is in state $\Psi(t)$, show that
$$\frac{d}{dt}\left\langle \hat{\Omega}\right\rangle = \frac{i}{\hbar}\left\langle \left[\hat{H}, \hat{\Omega}\right]\right\rangle + \left\langle \frac{\partial\hat{\Omega}}{\partial t}\right\rangle$$
where $\hat{H}$ is the hamiltonian operator. [Hint: If $f(t)$ and $g(t)$ are any two states, then prove that
$$\frac{d}{dt}\langle f(t), g(t)\rangle = \left\langle \frac{d}{dt}f(t), g(t)\right\rangle + \left\langle f(t), \frac{d}{dt}g(t)\right\rangle$$
using the properties of inner product and
$$\frac{d}{dt}\langle f(t), g(t)\rangle = \lim_{\Delta t\to 0}\frac{\langle f(t+\Delta t), g(t+\Delta t)\rangle - \langle f(t), g(t)\rangle}{\Delta t}.$$

6. If the hamiltonian operator of a system is given by $\hat{H} = \hat{P}^2/2m + V(\hat{X})$, then prove the Ehrenfest's theorem:
$$\begin{aligned}\frac{d}{dt}\left\langle \hat{X}\right\rangle &= \frac{1}{m}\left\langle \hat{P}\right\rangle \\ \frac{d}{dt}\left\langle \hat{P}\right\rangle &= \left\langle -V'(\hat{X})\right\rangle\end{aligned}$$
where V' is derivative of V wrt x. [Use the result of previous problem.]

7. Consider a quantum system consisting of a particle in a conservative force field. The energy spectrum is $\{E_1, E_2, \ldots\}$ with corresponding normalized stationary states $\{\phi_1, \phi_2, \ldots\}$. Let x_0 be an eigenvalue of the position operator with eigenvector ξ_{x_0}. Let $\alpha_1 = \langle \xi_{x_0}, \phi_1\rangle$ and $\alpha_2 = \langle \xi_{x_0}, \phi_2\rangle$. Let $\Psi(t)$ denote the state of the system at time t. Express, in terms of α_1, α_2 and eigenenergies, the answers to the following questions:

(a) If $\Psi(0) = \phi_1$, what is the probability density of finding the particle at x_0 at time t?

(b) If $\Psi(0) = \phi_2$, what is the probability density of finding the particle at x_0 at time t?

(c) If $\Psi(0) = (\phi_1 + \phi_2)/\sqrt{2}$, what is the probability density of finding the particle at x_0 at time t? What is the maximum probability density? And minimum?

8. Here is an example of time dependent Hamiltonian: An electron in an oscillating electric field is described by a Hamiltonian operator

$$\hat{H} = \frac{\hat{P}^2}{2m} - (eE_0 \cos \omega t)\, x$$

where E_0 is the amplitude of the electric field. Calculate $d\left\langle \hat{X} \right\rangle/dt$, and $d\left\langle \hat{P} \right\rangle/dt$.

9. The potential energy of a harmonic oscillator is given by $V(x) = m\omega^2 x^2/2$. Assuming that $\left\langle \hat{X} \right\rangle = \left\langle \hat{P} \right\rangle = 0$, find the lower limit to the expectation value of the Hamiltonian operator. [Hint: Use the uncertainty principle.]

10. The first excited state of the harmonic oscillator is given by

$$\psi_1(x) = \left(\frac{2\alpha}{\sqrt{\pi}} \right)^{1/2} (\alpha x)\, e^{-\alpha^2 x^2/2}.$$

Find ΔX and ΔP and check uncertainty principle. $\alpha^2 = m\omega/\hbar$.