1. Which of the following sets are vector spaces? (Assume usual function addition. Check only closure and existence of inverse.)

   (a) Piecewise continuous functions on $[a, b]$.
   (b) Twice differentiable functions on $[a, b]$.
   (c) Functions on $[0, a]$ satisfying the boundary conditions $f(0) = f(a)$.
   (d) Functions on $[0, a]$ satisfying the boundary conditions $f(0) = 0$ and $f(a) = 2$.
   (e) Functions satisfying the differential equation $y'' + y^2 = 0$.
   (f) Functions satisfying the differential equation $y'' + y = 0$.

2. Let $f_n : [0, \pi] \to \mathbb{R}$ such that $f_n(x) = \sin(nx)$ for $n = 1, 2, \ldots$. Show that the set $\{f_n | n = 1, 2, \ldots\}$ is orthogonal with respect to the inner product

$$\langle f_n, f_m \rangle = \int_0^\pi f_n(x) f_m(x) dx.$$

   Normalize these functions.

3. Prove Schwarz inequality,

$$\left| \int_a^b f^*(x) g(x) dx \right|^2 \leq \left[\int_a^b |f(x)|^2 \, dx \right] \left[\int_a^b |g(x)|^2 \, dx \right]$$

   for $f, g \in L_2([a, b])$. Use this identity to show that $L_2([a, b])$ is a vector space.

4. For what range of ν, is the function $f(x) = x^\nu$ in $L_2([0, 1])$. Assume ν to be real but not necessarily positive. For a specific case of $\nu = 1/2$, is f in $L_2([0, 1])$? What about $xf(x)$? And $(d/dx)f$?

5. Prove the following:

   (a) $(cA)^\dagger = c^* A^\dagger$
   (b) $(A + B)^\dagger = A^\dagger + B^\dagger$. Thus the sum of two hermitian operators is hermitian.
   (c) Show that $(AB)^\dagger = B^\dagger A^\dagger$. Thus the product of two hermitian operators is hermitian if they commute.
   (d) Hamiltonian operator

$$-\frac{\hbar^2}{2m} \hat{D}^2 + V(\hat{X})$$

   is hermitian. Here $V(\hat{X})$ is a function of the operator $\hat{X}$ and

$$\left(V(\hat{X}) f \right)(x) = V(x) f(x).$$

   Assume that the function $V(x)$ is real valued.

6. Let V be a finite dimensional inner product space. Let M_A be the matrix of an operator A with respect to an orthonormal basis. Show that

$$M_{A^\dagger} = \left[M_A^* \right]^T.$$

7. Show that the eigenvalues of hermitian operator are real. Also show that the eigenfunctions corresponding to distinct eigenvalues are orthogonal.

8. Let $W = \{f(\phi) \in L_2([0, 2\pi]) \mid f(0) = f(2\pi) \text{ and } f'(0) = f'(2\pi)\}$. Consider an operater $\hat{Q} = d^2/d\phi^2$ on W. Is $\hat{Q}$ hermitian? Find its eigenfunctions and eigenvalues.

9. The position operator $\hat{X} : L_2(\mathbb{R}) \to L_2(\mathbb{R})$ is defined as

$$\left(\hat{X} f\right)(x) = x f(x).$$

Find the eigenvalues and eigenfunctions of the position operator.

10. The matrix of an operator A on $\mathbb{R}^3$ is given by

$$\begin{bmatrix} a & 0 & b \\ 0 & c & 0 \\ a & 0 & a \end{bmatrix}.$$

Find the eigenvalues and eigenvectors.