Chapter 4

Special Functions

4.1 Legendre Polynomials

▷ Generating Function:

$$G(x,t) = \frac{1}{\sqrt{1-2xt+t^2}} \qquad |t| < 1,\ |x| \leq 1.$$

The Legendre polynomials are defined by

$$G(x,t) = \sum_{n=0}^{\infty} P_n(x) t^n$$

▷ Rodrigues Formula:

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} \left(x^2 - 1\right)^n$$

▷ Differential Equation:

$$\left[\left(1-x^2\right) \frac{d^2}{dx^2} - 2x\frac{d}{dx} + n\left(n+1\right)\right] P_n(x) = 0$$

▷ Orthogonality:

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{m,n}$$

▷ Recurrance Relations:

$$\begin{aligned} (n+1)P_{n+1}(x) &= (2n+1)xP_n(x) - nP_{n-1}(x) \\ (1-x^2)\frac{d}{dx}P_n(x) &= (n+1)xP_n(x) - (n+1)P_{n+1}(x) \end{aligned}$$

▷ Explicit Expressions:

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k)!}{2^n k!\,(n-k)!\,(n-2k)!} x^{n-2k}$$

First few polynomials:

$$\begin{aligned} P_0(x) &= 1 \\ P_1(x) &= x \\ P_2(x) &= \frac{1}{2}\left(3x^2-1\right) \\ P_3(x) &= \frac{1}{2}(5x^3-3x) \\ P_4(x) &= \frac{1}{8}\left(35x^4-30x^2+3\right) \end{aligned}$$

▷ Special Values:

$$\begin{aligned} P_n(1) &= 1 \\ P_n(-1) &= (-1)^n \\ P_n(0) &= \begin{cases} \frac{(-1)^{n/2}}{2^n}\begin{pmatrix} n \\ n/2 \end{pmatrix} & \text{even } n \\ 0 & \text{odd } n. \end{cases} \end{aligned}$$

4.2 Associated Legendre Functions

Normally, Legendre functions, P_ν^μ, are defined for arbitrary μ and ν, however here we will consider only a set of functions with integer ν and μ.

▷ Generating Function:

$$G_m(x,t) = \frac{(2m)!}{2^m m!}\frac{\left(1-x^2\right)^{m/2} t^m}{(1-2xt+t^2)^{m+1/2}} = \sum_{n=m}^{\infty} P_n^m(x) t^n$$

▷ Rodrigue's Formula:

$$P_n^m(x) = \left(1-x^2\right)^{m/2} \frac{d^m}{dx^m} P_n(x)$$

▷ Differential Equation:

$$\left[\left(1-x^2\right)\frac{d^2}{dx^2} - 2x\frac{d}{dx} + \left(n\,(n+1) - \frac{m^2}{(1-x^2)}\right)\right] P_n^m(x) = 0$$

▷ Orthogonality

$$\int_{-1}^{1} P_n^m(x) P_l^m(x) dx = \frac{2}{2n+1} \frac{(l+m)!}{(l-m)!} \delta_{n,l}$$

▷ Recurrance Relations:

$$\begin{aligned}
(2n+1)\, x P_n^m(x) &= (n-m+1) P_{n+1}^m(x) + (n+m)\, P_{n-1}^m(x) \\
(2n+1)\left(1-x^2\right)^{1/2} P_n^{m-1}(x) &= P_{l+1}^m(x) - P_{l-1}^m(x) \\
\left(1-x^2\right) \frac{d}{dx} P_n^m(x) &= -nx P_n^m(x) + (n+m) P_{n-1}^m(x) \\
\left(1-x^2\right) \frac{d}{dx} P_n^m(x) &= \left(1-x^2\right)^{1/2} P_n^{m+1}(x) - mx P_n^m(x)
\end{aligned}$$

▷ Explicit Expressions:

$$\begin{aligned}
P_1^1 &= \left(1-x^2\right)^{1/2} \\
P_2^1 &= 3\left(1-x^2\right)^{1/2} x \\
P_2^2 &= 3\left(1-x^2\right) \\
P_3^1 &= \frac{3}{2}\left(1-x^2\right)^{1/2}\left(5x^2-1\right) \\
P_3^2 &= 15x\left(1-x^2\right) \\
P_3^3 &= 15\left(1-x^2\right)^{3/2}
\end{aligned}$$

▷ Special Values:

$$P_n^m(1) = \frac{(2m)!}{2^m m!} \begin{pmatrix} -2m-1 \\ n-m \end{pmatrix}$$

where

$$\begin{pmatrix} -2m-1 \\ s \end{pmatrix} = \begin{cases} 1 & s = 0 \\ \frac{-m(-m-1)\cdots(1-s-m)}{s!} & s \geq 1 \end{cases}$$

4.3 Spherical Harmonics

▷ Definition:
For given l and $0 \leq m \leq l$

$$Y_{lm}(\theta, \phi) = (-1)^m \left[\frac{(2l+1)}{4\pi} \frac{(l-m)!}{(l+m)!}\right] P_l^m(\cos\theta) e^{im\phi}$$

and for $-l \leq m \leq 0$

$$\begin{aligned}
Y_{lm}(\theta, \phi) &= (-1)^m Y_{l,-m}^*(\theta, \phi) \\
&= \left[\frac{(2l+1)}{4\pi} \frac{(l+m)!}{(l-m)!}\right] P_l^{-m}(\cos\theta) e^{im\phi}
\end{aligned}$$

▷ Orthogonality:

$$\int Y^*_{l'm'}(\theta,\phi)Y_{lm}(\theta,\phi)d\Omega = \delta_{l,l'}\delta_{m,m'}$$

▷ Expansion of functions: If $f(\theta,\phi)$ are defined on S^2, then

$$f(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}Y_{lm}(\theta,\phi)$$

where

$$a_{lm} = \int Y^*_{lm}(\theta,\phi)f(\theta,\phi)d\Omega$$

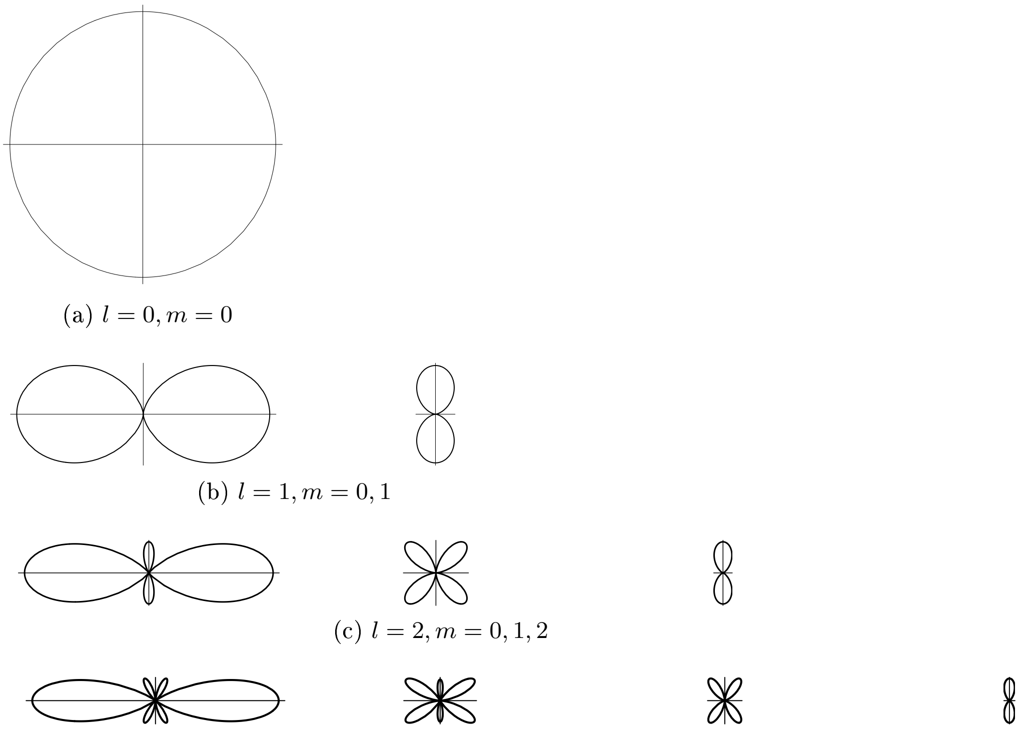
▷ Explicit Expressions:

$$\begin{aligned} Y_{l,0}(\theta,\phi) &= \left(\frac{2l+1}{4\pi}\right)^{1/2} P_l(\cos\theta) \\ Y_{l,l}(\theta,\phi) &= (-1)^l\left[\frac{2l+1}{4\pi}\frac{(2l)!}{4^l(l!)^2}\right]^{1/2}\sin^l\theta\, e^{il\phi} \end{aligned}$$

l	m	$Y_{lm}(\theta,\phi)$
0	0	$Y_{0,0} = \frac{1}{(4\pi)^{1/2}}$
1	0	$Y_{1,0} = \left(\frac{3}{4\pi}\right)^{1/2}\cos\theta$
	±1	$Y_{1,\pm1} = \mp\left(\frac{3}{8\pi}\right)^{1/2}\sin\theta e^{\pm i\phi}$
2	0	$Y_{2,0} = \left(\frac{5}{16\pi}\right)^{1/2}\left(3\cos^2\theta - 1\right)$
	±1	$Y_{2,\pm1} = \mp\left(\frac{15}{8\pi}\right)^{1/2}\sin\theta\cos\theta e^{\pm i\phi}$
	±2	$Y_{2,\pm2} = \left(\frac{15}{32\pi}\right)^{1/2}\sin^2\theta e^{\pm 2i\phi}$
3	0	$Y_{3,0} = \left(\frac{7}{16\pi}\right)^{1/2}\left(5\cos^3\theta - 3\cos\theta\right)$
	±1	$Y_{3,\pm1} = \mp\left(\frac{21}{64\pi}\right)^{1/2}\sin\theta\left(5\cos^2\theta - 1\right)e^{\pm i\phi}$
	±2	$Y_{3,\pm2} = \left(\frac{105}{32\pi}\right)^{1/2}\sin^2\theta\cos\theta e^{\pm 2i\phi}$
	±3	$Y_{3,\pm3} = \mp\left(\frac{35}{64\pi}\right)^{1/2}\sin^3\theta e^{\pm 3i\phi}$

▷ Parity: When $\mathbf{r}\to-\mathbf{r}$, $\theta\to\pi-\theta$ and $\phi\to\phi+\pi$.

$$Y_{l,m}(\pi-\theta,\phi+\pi) = (-1)^l Y_{lm}(\theta,\phi)$$



(a) $l = 0, m = 0$

(b) $l = 1, m = 0, 1$

(c) $l = 2, m = 0, 1, 2$

(d) $l = 3, m = 0, 1, 2, 3$

Figure 4.1: Polar Plots of $|Y_{lm}|^2$.

▷ Clearly,

$$
\begin{aligned}
\mathbf{L}^2 Y_{l.m}(\theta, \phi) &= l(l+1)\hbar^2 Y_{l.m}(\theta, \phi) \\
\mathbf{L}_z Y_{l.m}(\theta, \phi) &= m\hbar Y_{l.m}(\theta, \phi) \\
\mathbf{L}_- Y_{l.m}(\theta, \phi) &= \hbar\sqrt{l(l+1) - m(m-1)}\, Y_{l.m-1}(\theta, \phi) \\
\mathbf{L}_+ Y_{l.m}(\theta, \phi) &= \hbar\sqrt{l(l+1) - m(m+1)}\, Y_{l.m+1}(\theta, \phi) \\
\langle Y_{lm}, \mathbf{L}_x Y_{lm} \rangle &= 0 = \langle Y_{lm}, \mathbf{L}_x Y_{lm} \rangle \\
\left\langle Y_{lm}, \mathbf{L}_x^2 Y_{lm} \right\rangle &= \frac{1}{2} \left\langle Y_{lm}, \left(\mathbf{L}^2 - \mathbf{L}_z^2\right) Y_{lm} \right\rangle \\
&= \frac{\hbar^2}{2} \left[l(l+1) - m^2 \right]
\end{aligned}
$$

▷ Addition Theorem: Let $\mathbf{r}_1 = (r_1, \theta_1, \phi_1)$ and $\mathbf{r}_2 = (r_2, \theta_2, \phi_2)$. Let θ be the angle between vectors, $\mathbf{r}_1$ and $\mathbf{r}_2$. Then,

$$
P_l(\cos\theta) = \frac{4\pi}{2l+1} \sum_{m=-l}^{l} Y_{lm}^*(\theta_1, \phi_1) Y_{lm}(\theta_2, \phi_2).
$$

▷