1. Calculate the stationary states and energy eigenfunctions of a box with walls at $\pm L/2$.

2. Calculate $\langle X \rangle$, $\langle X^2 \rangle$, $\langle P \rangle$, and $\langle P^2 \rangle$ for the nth stationary state of the infinite square well. Check validity of the uncertainty principle.

   Solution:

   The average position is

$$\begin{aligned}
\langle X \rangle &= \frac{L}{2} \\
\langle X^2 \rangle &= \frac{L^2}{6}\left(2 - \frac{3}{\pi^2 n^2}\right) \\
\langle P \rangle &= 0 \\
\langle P^2 \rangle &= 2m\langle E \rangle = \frac{\hbar^2\pi^2}{L^2}n^2
\end{aligned}$$

   Note that as $n \to \infty$, $\langle X \rangle \to \langle X \rangle_{\text{classical}} = L^2/3$. Thus

$$\sigma_x = \frac{L}{2\sqrt{3}}\left(1 - \frac{6}{n^2\pi^2}\right)^{1/2}$$

   and $\sigma_p = \hbar\pi n/L$. Then,

$$\begin{aligned}
\sigma_x\sigma_p &= \frac{\hbar\pi n}{2\sqrt{3}}\left(1 - \frac{6}{n^2\pi^2}\right)^{1/2} \\
&= 0.568\hbar \qquad n = 1 \\
&\to \infty \qquad n \to \infty.
\end{aligned}$$

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3. Find the probability density function for momentum measurement if the particle is in the n^{th} stationary state of an infinite square well.

   Solution:
   Momentum space wave function is given by

$$\begin{aligned}
\phi(p) &= \frac{1}{\sqrt{2\pi\hbar}}\int_{-\infty}^{\infty} u_n(x)e^{-ipx/\hbar}dx \\
&= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{L}}\int_0^L \sin\left(\frac{p_n x}{\hbar}\right)e^{-ipx/\hbar}dx \\
&= \frac{1}{\sqrt{2\pi\hbar}}\sqrt{\frac{2}{L}}\frac{\hbar p_n}{p^2 - p_n^2}\left((-1)^n e^{ipL/\hbar} - 1\right)
\end{aligned}$$

   Thus the pdf for momentum measurement is

$$\begin{aligned}
\Pi(p) &= \phi^*(p)\phi(p) \\
&= \frac{2\pi n^2\hbar^3}{L^3}\left(1 - (-1)^n\cos\left(\frac{pL}{\hbar}\right)\right).
\end{aligned}$$

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4. A particle of mass m in the infinite square well (of width L) starts out in the left half of the well (at $t = 0$) and is equally likely to be found at any point in that region.

   (a) What is its initial wavefunction,$\Psi(x, 0)$? Assume that it is real.

   (b) What is the probability that a measurement of the energy would yield the value $\pi^2\hbar^2/2mL^2$?

Solution:

(a) The wave function is given by

$$\Psi(x,0) = \begin{cases} \sqrt{\frac{2}{L}} & x < \frac{L}{2} \\ 0 & x \geq \frac{L}{2}. \end{cases}$$

(b) Firstly, express

$$\Psi(x,0) = \sum_n c_n \phi_n(x),$$

where,

$$\begin{aligned} c_n &= \int_0^L \phi_n(x)\Psi(x,0)dx \\ &= \frac{4}{n\pi}\sin^2\left(\frac{n\pi}{4}\right) \end{aligned}$$

Probablity that the energy measurement will yield ϵ_n is $|c_n|^2$.

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5. A particle in infinite square well has the initial wave function

$$\Psi(x,0) = \begin{cases} Ax, & 0 \leq x \leq L/2, \\ A(L-x), & L/2 \leq x \leq L. \end{cases}$$

   (a) Sketch $\Psi(x, 0)$, and determine the constant A.

   (b) Find $\Psi(x, t)$.

   (c) How will you calculate the average energy, that is the expectation value of $\left\langle \hat{H} \right\rangle$. [The wave unction at $t = 0$ is not twice differentiable!]

Solution:

(a) $A = \sqrt{12/L^3}$.

(b) If $\Psi(x, 0) = \sum_n c_n\phi_n(x)$, then

$$c_n = \begin{cases} 0 & \text{even } n \\ \frac{4\sqrt{6}}{n^2\pi^2}(-1)^{(n-1)/2} & \text{odd } n. \end{cases}$$

Then

$$\Psi(x,t) = \sum_{\text{odd } n} \frac{4\sqrt{6}}{n^2\pi^2}(-1)^{(n-1)/2}\phi_n(x)\exp\left(-i\epsilon_n t/\hbar\right).$$

(c) The average energy,

$$\begin{aligned}\langle E\rangle &= \sum_n |c_n|^2 \epsilon_n \\ &= \frac{64}{\pi^4}\frac{\hbar^2\pi^2}{2mL^2}\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} \\ &= \frac{12\hbar^2}{2mL^2}\end{aligned}$$

Since, $\sum_{n=0}^{\infty}\frac{1}{(2n+1)^2} = \frac{\pi^2}{8}$.

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6. A particle in a infinite square well has its intial wave function given by

$$\Psi(x,0) = A\left(\sin\frac{\pi x}{a}\right)^5.$$

(a) Find A by normalizing the wavefunction.

(b) Find $\Psi(x,t)$.

(c) What is the probability that the energy measurement will yield ϵ_3?

Solution:

(a) First we write,

$$\begin{aligned}\Psi(x,0) &= A\left(\sin\frac{\pi x}{a}\right)^5 \\ &= A\frac{1}{16}\left[10\sin\left(\frac{\pi x}{a}\right) - 5\sin\left(\frac{3\pi x}{a}\right) + \sin\left(\frac{5\pi x}{a}\right)\right] \\ &= A\frac{1}{16}\sqrt{\frac{a}{2}}\left(10u_1(x) - 5u_3(x) + u_5(x)\right)\end{aligned}$$

By normalizing, we get $A = 16/\sqrt{63a}$.

(b) And,

$$\Psi(x,t) = \frac{1}{\sqrt{126}}\left(10u_1(x)e^{-i\epsilon_1 t/\hbar} - 5u_3(x)e^{-i\epsilon_3 t/\hbar} + u_5(x)e^{-i\epsilon_5 t/\hbar}\right)$$

(c) Probability that the energy measurement will yield ϵ_3 is $= |c_3|^2 = (-5/\sqrt{126})^2 = 25/126$.

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7. A particle in the infinite square well has as its initial wave function

$$\Psi(x,0) = \frac{1}{\sqrt{2}}\left(\phi_1(x) + \phi_2(x)\right)$$

where ϕ_n is the wavefunction of the n^{th}stationary state.

(a) Write down $\Psi(x,t)$ and $|\Psi(x,t)|^2$. Express later as a sinusoidal function of time. Use $\omega = \pi^2\hbar/2mL^2$.

(b) Find $\langle x\rangle$ as a function of t. It oscillates in time. What is the angular frequency? What is the amplitude?

(c) Compute $\langle p \rangle$ as a function of t. Check if it obeys Ehrenfest theorem, that is

$$\frac{d}{dt}\langle x \rangle = \frac{1}{m}\langle p \rangle .$$

Solution:

(a) Thus energy of nth stationary state is $\epsilon_n = n^2\hbar\omega$. Then

$$\Psi(x,t) = \frac{1}{\sqrt{2}}\left(\phi_1(x)e^{-i\omega t} + \phi_2(x)e^{-i4\omega t}\right)$$

and

$$|\Psi(x,t)|^2 = \frac{1}{2}\left[\phi_1^2(x) + \phi_2^2(x) + 2\phi_1(x)\phi_2(x)\cos(3\omega t)\right].$$

(b) The average value of position is given by

$$\begin{aligned}\langle x \rangle (t) &= \int_0^L |\Psi(x,t)|^2\, x dx \\ &= \frac{L}{2} - \frac{16L}{9\pi^2}\cos(3\omega t).\end{aligned}$$

The average particle position does a SHM in the box!! Also note that

$$\frac{d}{dt}\langle x \rangle (t) = \frac{8\hbar}{3L}\sin(3\omega t).$$

(c) Now average momentum is given by

$$\begin{aligned}\langle p \rangle (t) &= \int_0^L \Psi^*(x,t)\left[-i\hbar\frac{\partial}{\partial x}\Psi(x,t)\right] dx \\ &= \frac{8\hbar}{3L}\sin(3\omega t),\end{aligned}$$

Thus proving Ehrenfest theorem.

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8. Consider a system with a single particle moving in a conservative force field. The potential energy $V(x)$ of particle is bounded below, that is, $V(x) \geq V_{\text{min}}$ for all x. Show that there is no energy eigenstate with eigenvalue less than V_{min}. [Hint: If $E < V_{\text{min}}$, then ψ and its second derivative has same sign. Argue that such functions cannot be square integrable.]

Solution:
Schrodinger's TI equation gives

$$\frac{d^2}{dx^2}\psi(x) = \frac{2m}{\hbar^2}\left(V(x) - E\right)\psi(x)$$

Assume that $E < V_{min}$. Then ψ and ψ'' have same sign. Assume, that both are positive at a point x_0.

- Case 1: $\psi'(x_0) > 0$, then ψ will be monotonically increase for $x > x_0$ and the function will diverge as $x \to \infty$.
- Case 2: $\psi'(x_0) > 0$. Now, $\psi''(x_0) > 0 \implies \psi'$ is increasing and $\psi'(x_0) < 0 \implies \psi$ and ψ'' are decreasing functions. Now, suppose at x_1, ψ' crosses zero while ψ is still positive, then we have same situation as case 1.
  And if at x_1, ψ crosses zero and becomes negative, then all three functions ψ, ψ' and ψ'' have same sign and the functions will diverge as $x \to \infty$.
