1. Prove by mathematical induction that

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right).$$

2. A quantum particle of mass m is trapped in a square well of radius a. The potential of the well is given by

$$V(r) = \begin{cases} -V_0 & 0 \le r < a \\ 0 & r > a. \end{cases}$$

Let $E(< 0)$ denote the energy eigenvalue. Show that the radial part of the wave function is given by $j_l(q_1 r)$ inside the well and $k_l(q_2 r)$ outside, where l is angular momentum quantum number, $q_1^2 = 2m(E + V_0)/\hbar^2$ and $q_2^2 = -2mE/\hbar^2$. Write down the boundary condition at $r = a$.

3. Heat flows in a semi-infinite rectangular plate, the end $x = 0$, being kept at temperature θ_0 and the long edges $y = 0$ and $y = a$ at zero temperature. Find the temperature at a point in the plate.

4. Consider a region $V = \left\{(r, \theta, z) \,|\, a < r < b,\ 0 \le \theta \le \frac{\pi}{2}\right\}$. The value of electrostatic potential along the surface $r = b$ is $\theta\,(\pi/2 - \theta)$, and along all other surfaces the potential is zero. Find the electrostatic potential in V.

5. A circular wire of radius a charged with line density λ surrounds a grounded concentric spherical conductor of radius c. Determine the electrical charge density on the surface of the conductor.

6. Two halves of a long hollow conducting cylinder of inner radius b are separated by a small lengthwise gaps on each side, and are kept at different potentials V_1 and V_2. Show that the potential inside is given by

$$\Phi(s, \phi) = \frac{V_1 + V_2}{2} + \frac{V_1 - V_2}{2} \tan^{-1}\left(\frac{2bs}{b^2 - s^2}\cos\phi\right)$$

where ϕ is measured from a plane perpendicular to the plane through the gap. Calculate the surface charge density on each half of the cylinder.

7. A hollow right circular of radius b has its axis coincident with the z axis and its ends at $z = 0$ and $z = L$. The potential on the end faces is zero, while the potential on the cylindrical surface is given by

$$\Phi\,(b, \theta, z) = \begin{cases} V & \text{for } -\frac{\pi}{2} < \theta < \frac{\pi}{2} \\ -V & \text{for } \frac{\pi}{2} < \theta < \frac{3\pi}{2} \end{cases}$$

Find the potential inside the cylinder.

Tutorial 6

Q1. Given

$$j_n(x) = (-1)^n x^n \left(\frac{1}{x}\frac{d}{dx}\right)^n \left(\frac{\sin x}{x}\right) \quad \text{———} \quad \circledast$$

Clearly is true for $n=0$, i.e.

$$RHS = \frac{\sin x}{x} = j_0(x).$$

Assume $\circledast$ is true for $n=k$.

$$(-1)^{k+1} x^{k+1} \left(\frac{1}{x}\frac{d}{dx}\right)^{k+1} \left(\frac{\sin x}{x}\right)$$

$$= (-1)^{k+1} x^k \frac{d}{dx}\left[\left(\frac{1}{x}\frac{d}{dx}\right)^{k} \frac{\sin x}{x}\right]$$

$$\stackrel{\cdot}{=} (-1)^{k+1} x^k \frac{d}{dx}\left[(-1)^k x^{-k} j_k(x)\right]$$

$$= -x^k \frac{d}{dx}\left[x^{-k} j_k(x)\right] = j_{k+1}(x)$$ by recurrence relation.

Q2. Given potential -

$$V(r) = -V_0 \qquad 0 \le r < a$$
$$= 0 \qquad r > a$$

After substituting $\psi(\bar{r}) = R(r)\, Y_{\ell m}(\theta, \phi)$
we get, for $r < a$

$$R'' + \frac{2}{r} R' + \left[k^2 - \frac{\ell(\ell+1)}{r^2}\right] R = 0$$

where $k^2 = \dfrac{2m(E+V_0)}{\hbar^2} > 0$

This is the spherical Bessel Eq. The regular sol^n is spherical Bessel fn $j_\ell(kr)$.

For $r > k$, $q^2 = \frac{2mE}{\hbar^2} < 0$ Let $q^2 = -k^2$

The radial Eq. becomes

$$R'' + \frac{2}{r} R' - \left(k^2 + \frac{\ell(\ell+1)}{r^2} \right) R = 0$$

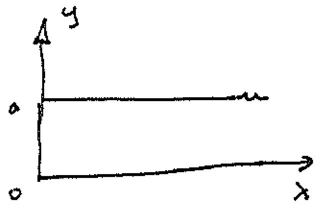
This is modified Bessel (Spherical) Eq. And the sol^n are called I_ℓ and k_ℓ. of which $I_\ell \to \infty$ as $r \to \infty$ hence the sol^n is k_ℓ outside.

The boundary cond. at $r = a$ is

$$j_\ell(ka) = k_\ell(ka)$$

$$j_\ell'(ka) = k_\ell'(k'a)$$

Tutorial 6



Q3. After Separation of variables:

$$\phi(x,y) = X(x)\, Y(y)$$

then

$$X(x) = Ae^{kx} + Be^{-kx}$$

$$Y(y) = C\sin(ky) + D\cos(ky)$$

Since $Y(0) = Y(a) = 0 \Rightarrow k = \frac{n\pi}{a}$ and $D = 0$. n, integer $1, 2, ..$

Thus $$Y(y) = C\sin\left(\frac{n\pi y}{a}\right).$$

since $X(x) \to 0$ as $x \to \infty \Rightarrow A = 0$.

$$\therefore \quad \phi(x,y) = \sum_n A_n e^{-n\pi x/a} \sin\left(\frac{n\pi y}{a}\right)$$

Since $\phi(0, y) = \theta_0$ (Given temp)

$$\Rightarrow \quad \theta_0 = \sum_n A_n \sin\left(\frac{n\pi y}{a}\right)$$

$$\therefore \quad \frac{a}{2} \cdot A_n = \int_0^a \theta_0 \sin\left(\frac{n\pi y}{a}\right) dy = -\theta_0 \left(\cos(n\pi) - 1\right) \cdot \frac{a}{n\pi}.$$

$$A_n = +\frac{4\theta_0}{n\pi}$$

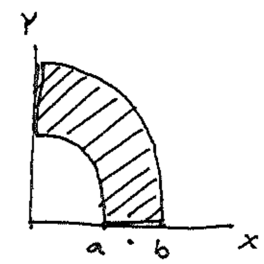
$\therefore$ Temperature in the plate is given by

$$\phi(x,y) = \frac{4\theta_0}{\pi} \sum_n \frac{1}{n}\, e^{-n\pi x/a} \sin(n\pi y/a).$$

Tutorial 6

Q4. After separation of variable

$$\phi(r,\theta) = R(r)\,T(\theta)$$



Then

$$R(r) = A r^{m} + B r^{-m}$$

$$T(\theta) = C \sin(m\theta) + D \cos(m\theta).$$

BC are $\phi(r,0) = 0$ and $\phi(r,\frac{\pi}{2}) = 0$

$$\Rightarrow T(0) = T(\pi/2) = 0$$

Now $\Rightarrow$ $T(0) \neq 0 \Rightarrow D = 0$ and $T(\pi/2) = 0 \Rightarrow m = 2n$

$n = \cancel{0}, 1, 2, \dots$

$\therefore$ $T(\theta) = c \sin(2n\theta)$ $\quad n = 1, 2, \dots$

$\therefore$ $\phi(r,\theta) = \sum_n \left(A_n r^{2n} + B_n r^{-2n}\right) \sin(2n\theta)$

Since $\phi(a,\theta) = 0.$

$$\Rightarrow A_n = -B_n/a^{4n}$$

and $\phi(b,\theta) = \theta\cdot(\pi/2 - \theta)$

$$\Rightarrow \sum_n \left(A_n b^{2n} + B_n b^{-2n}\right) \sin(2n\theta) = \theta(\pi/2 - \theta)$$

$$\Rightarrow B_n \frac{\pi}{4}\left(-\frac{b^{2n}}{a^{4n}} + \frac{1}{b^{2n}}\right) = \int_0^{\pi/2} \sin(2n\theta)\,\theta\,(\pi/2 - \theta)\,d\theta$$

$$= B_n \frac{\pi}{4} \frac{a^{4n} - b^{4n}}{a^{4n} b^{2n}} = -\frac{-2 + 2(-1)^n + n\pi \sin(n\pi)}{8n^3}$$

$\therefore$ $B_n = \frac{4}{\pi} \frac{a^{4n} b^{2n}}{(b^{4n} - a^{4n})} \cdot \frac{(-4\quad)}{8n^3}$ $\quad n$ odd.

$= -\frac{2}{\pi} \frac{a^{4n} b^{2n}}{(b^{4n} - a^{4n})}$ $\quad$ ~~n even~~

$= 0$ $\quad n$ ~~odd~~ even

Q4.

$$\phi(r,\theta) = \frac{\cancel{4}2}{\pi} \sum_{n,\,\mathrm{odd}} \left[\frac{a^{4n} b^{2n}}{a^{4n} - b^{4n}} \right] \left[- \frac{r^{2n}}{a^{4n}} + \frac{1}{r^{2n}} \right] \sin(2n\theta)$$

Tut 6

Q5. (B) By method of images

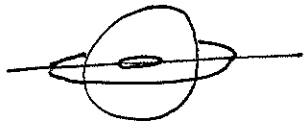


Image of the ring will be another ring inside the sphere of radius a' and charge density λ'.

Clearly, $a' = c^2/a$. To find λ', image of an elementary charge $(\lambda a\, d\theta)$ be $(\lambda' a'\, d\theta)$.

Then
$$(\lambda' a'\, d\theta) = -\frac{c}{a}(\lambda a\, d\theta)$$

$$\Rightarrow \quad \lambda' = -\frac{\lambda a}{c}.$$

Thus the potential on z axis due to two rings will be

$$\phi = \frac{2\pi\lambda a}{\sqrt{a^2+z^2}} + \frac{2\pi\lambda' a'}{\sqrt{a'^2+z^2}}$$

$$= (2\pi\lambda a)\left[\frac{1}{\sqrt{a^2+z^2}} - \frac{c}{\sqrt{c^4+a^2z^2}}\right]$$

Thus for $c < z < a$

$$\phi(z) = 2\pi\lambda a \sum_{n=0}^{\infty}\left[\frac{1}{a}\alpha_n \left(\frac{z}{a}\right)^{2n} - \frac{c}{az}\alpha_n\left(\frac{c^2}{az}\right)^{2n}\right]$$

$$= 2\pi\lambda \cancel{a} \sum_n \frac{\alpha_n}{a^{2n}}\left[z^{2n} - \frac{c^{4n+1}}{z^{2n+1}}\right], \quad \alpha_n = \frac{(-1)^n(2n-1)!!}{2^n\, n!}$$

$\therefore$ For $c < r < a$ (By extension)

$$\phi(r,\theta) = 2\pi\lambda \sum_n \frac{\alpha_n}{a^{2n}}\left[r^{2n} - \frac{c^{4n+1}}{r^{2n+1}}\right] P_{2n}(\cos\theta).$$

Similarly $c < a < r$

$$\phi(r,\theta) = 2\pi\lambda \sum_n \frac{\alpha_n}{r^{2n+1}}\left(\frac{a^{4n+1} - c^{4n+1}}{a^{2n}}\right) P_{2n}(\cos\theta)$$

Q5 (b) by variable separation, by linearity (superposition)

$$\phi(r,\theta) = \phi \text{ due to ring} + \phi \text{ due to sphere.}$$

For $c < r < a$

$$\phi_{ring}(r,\theta) = 2\pi\lambda \sum_n \alpha_n \left(\frac{r}{a}\right)^{2n} P_{2n}(\cos\theta)$$

Assume $\phi_{sphere}(r,\theta) = \sum_n \frac{B_n}{r^{n+1}} P_n(\cos\theta)$

$$\therefore \quad \phi(r,\theta) = \sum_n \left(\frac{B_n}{r^{n+1}} + \underbrace{\frac{2\pi\lambda\,\alpha_{n/2}}{a^n} r^n}_{\text{even } n \text{ only.}} \right) P_n(\cos\theta)$$

Since $\phi(c,\theta) = 0 \Rightarrow$

$$\frac{B_n}{c^{n+1}} = -\frac{2\pi\lambda\,\alpha_{n/2}}{a^n} c^n \qquad \text{for even } n.$$

$$= 0 \qquad \text{for odd } n.$$

$$\Rightarrow \quad B_{2n} = -\frac{2\pi\lambda\,\alpha_n}{a^{2n}} c^{4n+1}$$

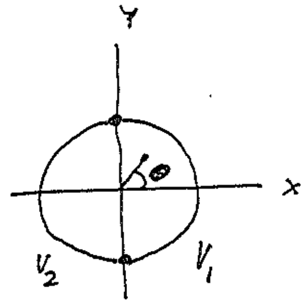
$$\Rightarrow \quad \phi(r,\theta) = 2\pi\lambda \sum_n \frac{\alpha_n}{a^{2n}} \left(r^{2n} - \frac{c^{4n+1}}{r^{2n+1}} \right) P_{2n}(\cos\theta).$$

Can calculate charge on the surface.

$$\sigma = -\epsilon_0 \left.\frac{\partial\phi}{\partial r}\right|_{r=c} = 2\pi\lambda \sum_n \frac{\alpha_n}{a^{2n}} \left(2n\, c^{2n-1} + (2n+1)\, c^{2n-1} \right) P_{2n}(\cos\theta)$$

$$= 2\pi\lambda \sum_n \frac{\alpha_n}{a^{2n}} (4n+1)\, c^{2n-1} P_{2n}(\cos\theta).$$

Tutorial 6



Q6. After variable separation

$$\phi(s,\theta) = R(s)\, T(\theta)$$

Thus $\quad R(s) = A s^{m} + B s^{-m}$

and $\quad T(\theta) = C \sin(m\theta) + D \cos(m\theta)$

~~Unique~~ Single-valued-ness of $T \Rightarrow m$ must be an Integer. $0, 1, \cdots$.

[Note: From symmetry, $C = 0$, but can be explicitly shown].

And $R(s) < \infty$ for $s \to 0 \Rightarrow B = 0$.

$$\therefore \quad \phi(s,\theta) = \sum_{m=0}^{\infty} \left[C_m \sin(m\theta) + D_m \cos(m\theta) \right] \cdot s^{m}$$

Finally $\quad \phi(b,\theta) = V_1 \quad$ if $\quad \theta \in [-\pi/2, \pi/2]$

$\qquad\qquad\qquad\quad = V_2 \quad$ if $\quad \theta \in [\pi/2, 3\pi/2]$

$$\therefore \quad \pi b^{m} C_m = V_1 \int_{-\pi/2}^{\pi/2} \sin(m\theta)\, d\theta + V_2 \int_{\pi/2}^{3\pi/2} \sin(m\theta)\, d\theta = 0$$

$$\text{and} \quad \pi b^{m} D_m = V_1 \int_{-\pi/2}^{\pi/2} \cos(m\theta)\, d\theta + V_2 \int_{\pi/2}^{3\pi/2} \cos(m\theta)\, d\theta =$$

$$\doteq 2\, \frac{V_1 - V_2}{m}$$

$$\Rightarrow \quad D_m = \frac{2(V_1 - V_2)}{m \pi b^{m}} \sin\left(\frac{m\pi}{2}\right) \qquad m \neq 0, \quad \text{odd}$$

$$\qquad = 0 \qquad m \neq 0 \quad \text{even}$$

and. $\quad D_0 = \dfrac{V_1 + V_2}{2}$

$$\therefore \quad \phi(s,\theta) = \frac{V_1 + V_2}{2} + \frac{2(V_1 - V_2)}{\pi} \sum_{m=0}^{\infty} \frac{(-1)^{m}}{(2m+1)} \left(\frac{s}{b}\right)^{2m+1} \cos((2n+1)\theta)$$

$$= \frac{V_1 + V_2}{2} + \frac{2(V_1 - V_2)}{\pi} \operatorname{Im} \sum_{m=0}^{\infty} \frac{i\, i^{2m}}{(2m+1)} \left(\frac{s}{b}\right)^{m} e^{i(2n+1)\theta}$$

$$= \frac{V_1 + V_2}{2} + \frac{2(V_1 - V_2)}{\pi} \operatorname{Im} \left[\ln\left(\frac{1 + \frac{s}{b} e^{i(\theta + \pi/2)}}{1 - s/b\, e^{+i(\theta + \frac{\pi}{2})}} \right) \right]$$

$$= \frac{V_1 + V_2}{2} + \frac{2(V_1 - V_2)}{\pi} \tan^{-1}\left(\frac{2sb}{b^2 - s^2} \cos\theta \right)$$

Tutorial 6.

Q 7. In cylindrical co-ordinates, Laplace Eq.

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \Phi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} + \frac{\partial^2}{\partial z^2} \right)\Phi = 0$$

Using Variable separation technique

$$\Phi(r,\theta,z) = R(r)\,T(\theta)\,Z(z)$$

Let $\frac{d^2}{dz^2}Z(z) = -k^2 Z(z)$ and $\frac{\partial^2}{\partial\theta^2}T(\theta) = -m^2 T(\theta)$

Then radial Eq. turns out to be

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} - \left(\frac{m^2}{r^2} + k^2\right)R = 0$$

$$\Rightarrow \quad r^2R'' + rR' - (k^2r^2 + m^2)R = 0 \qquad (1)$$

This eq. is called as modified Bessel Eq. (Note $-$ve sign with $\underline{k^2r^2}$.

T must be single valued $\Rightarrow$ m is an integer

and $Z(0) = Z(L) = 0 \Rightarrow Z = \sin\left(\frac{n\pi z}{L}\right)$, $n = 1, 2, \dots$

$$T = A\sin(m\theta) + B\cos(m\theta)$$

Two sol^n to (1) are denoted by I_ν and K_ν.

I_m are regular at $r \to 0$ and diverge as $r \to \infty$

K_m are singular at $r \to 0$ and $\to 0$ as $r \to \infty$.

The series definitions are

$$I_\nu(x) = \sum_{n=0} \frac{1}{(n!)(n+\nu)!}\left(\frac{x}{2}\right)^{2n+\nu}$$

$$K_\nu(x) = \frac{\pi}{2}\left[\frac{I_{-\nu}(x) - I_\nu(x)}{\sin(\nu\pi)}\right]$$

Since, in present problem, $0 \le r \le b$, clearly

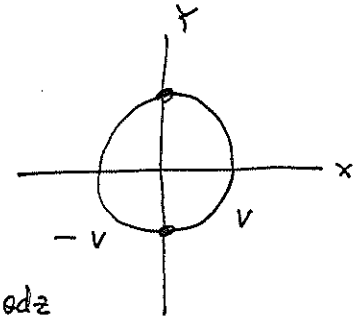
$$R(r) = I_m\left(\frac{n\pi}{L}r\right)$$

$\therefore$

$$\Phi = \sum_{m,n} I_m\left(\frac{n\pi}{L}r\right)\cdot \sin\left(\frac{n\pi z}{L}\right)\left\{A_{m,n}\sin(m\theta) + B_{m,n}\cos(m\theta)\right\}$$

Since $\Phi(b,\theta,z) = V \qquad \theta \in -\frac{\pi}{2} \text{ to } \frac{\pi}{2}$

$= -V \qquad \frac{\pi}{2} < \theta < \frac{3\pi}{2}$



$$\therefore \quad A_{m,n} = \frac{2}{\pi L I_m}\cdot\int \phi(b,\theta,z)\cdot \sin\left(\frac{n\pi z}{L}\right)\sin(m\theta)\,d\theta\,dz$$

and $$B_{m,n} = \frac{2}{\pi L I_m}\int \phi(b,\theta,z)\sin\left(\frac{n\pi z}{L}\right)\cos(m\theta)\,d\theta\,dz$$

$\therefore \quad A_{m,n} = 0 \qquad \forall\ m, n.$

$$B_{m,n} = \frac{2V}{\pi L I_m(kb)}\left[\int_0^L \sin\left(\frac{n\pi z}{L}\right)dz\right]\left[\int_{-\pi/2}^{\pi/2}\cos(m\theta)\,d\theta - \int_{\pi/2}^{3\pi/2}\cos(m\theta)\,d\theta\right]$$

$$= \frac{2V}{\pi L I_m(kb)}\cdot\left[\frac{L}{n\pi}\left(1-\cos n\pi\right)\right]\left[\frac{4}{m}\sin\left(\frac{m\pi}{2}\right)\right] \qquad \text{odd } m,\ \text{all } n.$$

$$= \frac{16(-1)^m V}{\pi^2 mn I_m(kb)}. \qquad \text{odd } m,\ \text{odd } n$$

$$= 0 \qquad \text{otherwise}$$

$$\therefore \quad \phi(r,\theta,z) = \sum_{\substack{m,n\\(\text{odd})}} I_m\left(\frac{n\pi r}{L}\right)\sin\left(\frac{n\pi z}{L}\right)\cos(m\theta)\cdot\frac{16(-1)^m V}{\pi^2 mn\, I_m\left(\frac{n\pi b}{L}\right)}$$