Notes on Differential Equations

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2009

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1. Differential Equations : Review of First order Equations

1.1 Intro : Definition : Let $f(x,y)$ be a function defined for all real x in some interval I and for ~~all~~ complex y in some set S. Let $\phi(x)$ be a real valued function defined over interval I s.t.

(i) $\phi(x) \in S$

(ii) $\phi'(x) = f(x, \phi(x))$.

Equation (ii) is called an ~~[illegible]~~ ordinary differential equation of first order.

Remark : A typical problem is that the ~~equat~~ differential equation is known and $\phi(x)$ needs to be found. The function ϕ is called a solution of the d.e.. $\phi(x)$ may not exist at all / or many soln.

Example : $dP/dt = F(t)$ Newton's Equation

Example : $dN/dt = +\lambda N$ Radioactive Decay

Example : $dT/dt = k(T-T_0)$ Newton's Law of cooling

Remark : x – independent variable, y – dependent variable

Ordinary – derivatives are not partial

First order – Order (highest) of derivative involved.

Remark : General form of DE of order n

$$F(x, y, y', y'', \dots, y^{(n)}) = 0$$

Example : $y'' + y = 0$ Oscillations.

Example : $m\ddot{x} = f(t, x, \dot{x}) = 0$

Remark : 1. Solutions can be obtained in a closed form ~~is~~ or numerically.

2. DE may have several solns eg. $y' = f(x)$: has infinite soln. if f is integrable. We may want to write down all soln. or specific soln. satisfying some cond.

Main types : initial value problems (at one point), boundary value problems.

Example : $y' = \cos x$, $y(0) = 0$, $y = \sin x + c \Rightarrow c = 0$

~~$y' = \cos x$, $y(0) = 1$, $y(\pi) = 1$~~

Example : $y' = -y$, $y(0) = 1$, ~~$y(2\pi) = -1$~~ $y(\pi/2) = 1$

$y = \cos x + \sin x$

Remark : Many times, we may not be able to find solutions, though they exist.

1.2 Some General Methods of solving FODE

1.2.1 Seperation of variables

If a DE can be written as

$$M(x)\,dx + N(y)\,dy = 0$$

Soln. $\int M(x)dx + \int N(y)dy = c$

Example : $e^{x^2}y' + y^2x = 0$

$\Rightarrow dy/y^2 + x\,dx/e^{x^2} = 0$

Soln. $-1/y - \frac{1}{2}e^{-x^2} = c$

$ye^{-x^2} + 2 + 2cy = 0$

Example $x^2(y+1)dx + y^2(x-1)dy = 0$

$\Rightarrow x^2dx/(x-1) + y^2dy/(y+1) = 0$

$\Rightarrow [(x+1) + 1/(x-1)]dx + [(y-1) + \frac{1}{(y+1)}]dy = 0$

$\Rightarrow (x^2+y^2)/2 + (x-y) + \ln[(x-1)(y+1)] = c$

1.2.2 Exact Differentials

Definition: A DE of the form $Mdx + Ndy = 0$ is called exact if $\exists$ a dif. function $u(x,y)$ s.t. $du = Mdx + Ndy$. The solution to the DE is $u(x,y) = c$

Example : $y\cos(xy)dx + x\cos(xy)dy = 0$ is exact since $u(x,y) = \sin(xy)$

Theorem : The DE $Mdx + Ndy = 0$ is an exact equation iff M and N have continuous partial derivatives and $\partial M/\partial y = \partial N/\partial x$

<u>Proof</u>: ⓐ If $Mdx + Ndy = 0$ is exact $\exists\ u(x,y)$

$\Rightarrow \partial u/\partial x = M, \ \partial u/\partial y = N$

$\Rightarrow \partial M/\partial y = \partial^2 u/\partial y\partial x$ and $\partial N/\partial x = \partial^2 u/\partial x\partial y$

$\Rightarrow \partial M/\partial y = \partial N/\partial x$ since cont. derivatives

ⓑ let $\phi(x,y) = \int M dx$ or $\partial\phi/\partial x = M$

$\Rightarrow \partial N/\partial x = \partial M/\partial y = \partial^2\phi/\partial y\partial x = \partial^2\phi/\partial x\partial y$

$\Rightarrow N = \partial\phi/\partial y + f(y)$ let $F(y) = \int f(y)dy$

$\Rightarrow \partial/\partial x[\phi(x,y) + F(y)]\,dx + \frac{\partial}{\partial y}[\phi(x,y) + F(y)]\,dy = 0$

$\Rightarrow u(x,y) = \phi(x,y) + F(y)$

Examples $(\cos y + y\cos x)dx + (\sin x - x\sin y)dy = 0$

$\partial M/\partial y = -\sin y + \cos x = \partial N/\partial y$

$u(x,y) = \int Mdx + \int(\text{terms of } N \text{ containing } y \text{ only})\,dy$

$= x\cos y + y\sin x + 0$

Example : Some standard differentials

$d\left(\frac{1}{2}(x^2+y^2)\right) = x\,dx + y\,dy$

$d(xy) = y\,dx + x\,dy$

$d(x/y) = (y\,dx - x\,dy)/y^2$

$d(\log(x/y)) = (y\,dx - x\,dy)/xy$

$d(\tan^{-1}(x/y)) = (y\,dx - x\,dy)/(x^2+y^2)$

Integrating Factors. If $M\,dx + N\,dy = 0$ is not exact, sometimes can be made exact by multiplying by a function $\lambda(x,y)$, called integrating factor.

Example : $y\,dx - x\,dy = 0$ is not exact but by multiplying $(1/xy)$ it becomes exact. Several IF may exist.

Theorem : If $M\,dx + N\,dy = 0$ is not exact but

(i) $\frac{1}{N}\left(\partial M/\partial y - \partial N/\partial x\right) = f(x)$ is function of x alone then

$IF = e^{\int f(x)dx}$

(ii) $\frac{1}{M}\left(\partial N/\partial x - \partial M/\partial y\right) = g(y)$ is a function of y alone then

$IF = e^{\int g(y)dy}$

(iii) $Mx + Ny \neq 0$ and The eq. is homogeneous then

$IF = 1/(Mx+Ny)$

Proof: (i) show that $e^{\int f(x)dx}(M\,dx + N\,dy)$ is exact.

Example: $(3x^2y^4 + 2xy)dx + (2x^3y^3 - x^2)dy = 0$, category (i) $IF = 1/y^2$

Example: $(x^2y - 2xy^2)dx + (3x^2y - x^3)dy = 0$ homogeneous $Mx+Ny \neq 0$

$IF = 1/x^2y^2$

1.2.3 Bernoulli's Equation

$$dy/dx + p(x)y = Q(x)y^n$$

put $z = y^{1-n}$ $\quad dz/dx = y^{-n}(1-n)\,dy/dx$

$$\frac{y^n}{(1-n)} \cdot dz/dx + p(x)\cancel{y^n}y = Q(x)y^n$$

$$dz/dx + (1-n)p(x)z = (1-n)Q(x)$$

$$dz/dx + p_1^*(x)z = Q_1(x)$$

Soln. from Linear Equations is given by

$$z = e^{-\int p(x)dx}\left[\int Q(x)e^{\int p(x)dx}dx + c\right]$$

Example: $dy/dx - xy = x^3y^2$

1.3 Linear First order Differential Equations

1 A General Linear ODE of First order is given by
$y' + p(x)y = Q(x)$. why its called linear. Let $D = \frac{d}{dx}$

$$(D+p(x))(y_1 + c y_2) = (D+p(x))y_1 + c(D+p(x))y_2$$

If $Q(x) = 0 \ \forall x$ then the DE is called homogeneous else nonhomogen.

1.3.1 The equation $y' + ay = 0$, a is constant
the solution is $y = Ae^{-ax}$, where A is any complex no.

Example: (Radioactive Decay)

$$dN/dt = -\lambda N \Rightarrow dN/dt + \lambda N = 0, \quad N(t=0) = N_0$$

$$N = N_0 e^{-\lambda t}$$, Usually posed as initial value problem.

1.3.2 The equation $y' + ay = b(x)$

write $dy + [ay - b(x)]dx = 0$ then $M = ay - b(x)$, $N = 1$

and $\partial N/\partial x = 0$, $\partial M/\partial y = a$

then $\frac{1}{N}(\partial M/\partial y - \partial N/\partial x) = a$ is a function of x only

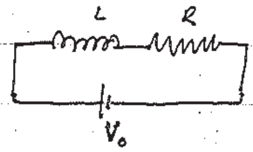
$\Rightarrow$ $IF = e^{ax}$

$\Rightarrow$ $e^{ax}dy + e^{ax}[ay - b(x)]dx = 0$

$\Rightarrow$ Soln is $ye^{ax} - \int e^{ax}b(x)\,dx = c$ $\quad$ C is any complex no.

$\Rightarrow$ $y = e^{-ax}\int e^{ax}b(x)dx + Ce^{-ax}$

Example: (LR Circuit) A ckt is switched on at $t=0$.

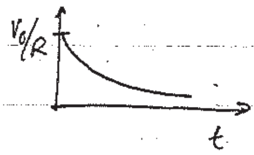


$V_0 = L\,di/dt + iR$

$\Rightarrow di/dt + i(R/L) = V_0/L$

$\Rightarrow i = V_0/R + Ce^{-tR/L}$

$\Rightarrow i = V_0/R\,(1 - e^{-t/\tau})$ $\quad$ $\tau = L/R$



1.3.3 General Linear Equation

$$y' + p(x)y = Q(x)$$

Notice that $(e^{I(x)}y)' = e^{I(x)}(y' + I'(x)y)$

Let $I'(x) = p(x)$ then

$$e^{I(x)}(y' + p(x)y) = (e^{I(x)}y)' = e^{I(x)}Q(x)$$

$\Rightarrow$ $e^{I(x)}y = \int e^{I(x)}Q(x)\,dx + C$ $\quad$ C is any complex no.

$\Rightarrow$ $y = e^{-I(x)}\int e^{I(x)}Q(x)dx + Ce^{-I(x)}$

where $I(x) = \int p(x)dx$

Example: (Bernoulli's Equation)

$$y' + p(x)y = Q(x)y^m$$

put $z = y^{1-m}$

$$z' + (1-m)p(x)z = (1-m)Q(x)$$

$$z = y^{1-m} = e^{-I(x)} \int e^{I(x)} (1-m) Q(x)\,dx + c\, e^{-I(x)}$$

$$I(x) = (1-m)\int p(x)\,dx.$$

Example: (Viscosity) Force proprtional to v^2

$$m\,dv/dt = -\alpha v^2 \qquad \text{put } v^{-1} = z$$

$$dz/dt = +(\alpha/m)$$

$$z = 1/v = (\alpha/m)t + c$$

$$v = 1/(\alpha/m\, t + c) \qquad \text{if} \quad t = 0 \quad v = v_0$$

Example: $dy/dx + 2xy/(1+x^2) = \cot x/(1+x^2)$

$$p(x) = 2x/(1+x^2) \qquad I(x) = \int \frac{2x}{1+x^2}\,dx = \ln(1+x^2)$$

$$IF = e^{I(x)} = (1+x^2)$$

$$y(1+x^2) = \int (1+x^2)\cdot \cot x/(1+x^2)\,dx + c$$

$$= \log|\sin x| + c$$

$$y = (\log|\sin x| + c)/(1+x^2)$$

Example $(x^2y^3 + 2xy)dy = dx$

$$dx/dy - 2y\cdot x = y^3x^2$$

put $z = 1/x$

$$dz/dy + 2yz = -y^3$$

$$p(y) = 2y \qquad I(y) = \int 2y\,dy = y^2$$

$$If = e^{y^2}$$

$$z e^{y^2} = -\int y^3 e^{y^2} dy + c$$

$$\int y^3 e^{y^2} dy = \frac{1}{2}\int u e^u du = (u\cdot e^u - e^u)\frac{1}{2} = (y^2 \cancel{e^{y^2}} - 1)e^{y^2}/2$$

$$z e^{y^2} = -\frac{1}{2} e^{y^2}(y^2 - 1) + c$$

$$\frac{1}{x} = z = (1-y^2)/2 + ce^{-y^2}$$

$$x = [(1-y^2)/2 + ce^{-y^2}]^{-1}$$

2. Second order Linear Differential Equations

2.1 Some general properties

Definition: A DE of the form

$$y'' + P(x)\, y' + Q(x)\, y = R(x)$$

where $P(x), Q(x), R(x)$ are real valued continuous function on I.

Remark: Homogeneous if $R(x) = 0 \quad \forall\, x \in I$.

Theorem: Let P, Q, R be cont. real functions on I. Let $x_0 \in I$ and α, β arbitrary constants. Then there exist a unique solution $y(x)$ satisfying DE

$$y'' + P(x)\, y' + Q(x)\, y = R(x)$$

$\forall\, x \in I$ and initial conditions $y(x_0) = \alpha$ and $y'(x_0) = \beta$.

Remark: Proof is not included. Initial conditions as against the boundary conditions, in which case sol^n cannot be guaranteed.

Definition: For given functions y_1 and y_2, $C_1 y_1 + C_2 y_2$, where C_1, C_2 are real nos., is called LC of y_1 and y_2

Definition: Two functions y_1 and y_2 are LI on a interval I if $C_1 y_1 + C_2 y_2 = 0 \Rightarrow C_1 = 0$ and $C_2 = 0$.

y_1 and y_2 are Linearly Dependent if they are not LI

Example: Let $y_1 = x \quad y_2 = x^2 \quad I = [0,1]$

Let $C_1 y_1 + C_2 y_2 = 0 \quad \forall\, x \in I$

$C_1 \cdot 1/4 + C_2 (1/16) = 0 \quad x = 1/4$

$4C_1 + C_2 = 0$ ——— ①

$2C_1 + C_2 = 0 \quad x = 1/2$

$\Rightarrow C_1 = C_2 = 0$

Example $\Rightarrow$ $y_1 = \sin x \quad y_2 = \cos x$ are LI on $I = [0, 2\pi]$.

Theorem: Let $y_1(x)$ and $y_2(x)$ be two solutions of homogeneous linear DE

$$y'' + P(x)\, y' + Q(x)\, y = 0$$

then $C_1 y_1(x) + C_2 y_2(x)$ ~~are~~ is also a solution of the equation.

Proof: Straightforward substitution.

Definition: A general solution y_g of a diff. Eq.

$$y'' + P(x)\, y' + Q(x) y = R(x)$$

is such that any solution ~~of~~ y of this diff. Eq. can be obtained from y_g.

Example: (Illustrate idea of a general solution)

Consider $y'' + 4y = 3\sin x$

One solution is $y_1 = \sin 2x + \sin x$.

another solution is $y_2 = \cos 2x + \sin x$

General Solution is $y_g = C_1 \sin 2x + C_2 \cos 2x + \sin x$

Any other solution can be obtained from y_g by choosing C_1 and C_2.

Theorem : Let functions P, Q be cont on I, then DE $L(y) = \underbrace{y'' + P(x)y' + Q(x)y}_{=0}$ always posseses two LI solutions on I.

Proof : let y_1 be solution of $L(y) = 0$, $y_1(x_0) = 1$, $y_1'(x_0) = 0$

" y_2 " $L(y) = 0$, $y_2(x_0) = 0$, $y_2'(x_0) = 1$

Existence of y_1 and y_2 is guaranteed. Let C_1 and C_2 be such that $\circledast\ C_1 y_1 + C_2 y_2 = 0 \quad \forall x \in I$

in particular $C_1 y_1(x_0) + C_2 y_2(x_0) = 0$ $\qquad \circledast$ then $C_1 y_1' + C_2 y_2' = 0$

and $C_1 y_1'(x_0) + C_2 y_2'(x_0) = 0$

$\Rightarrow C_1 = 0$ and $C_2 = 0$

$\Rightarrow y_1$ and y_2 are LI.

Definition : If y_1, y_2 are two diff. functions on I, ~~then the~~ determinant

$$W(y_1, y_2) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix}$$

is called the Wronskian of y_1 and y_2.

Theorem : If y_1 and y_2 are two solutions of $y'' + P(x)y' + Q(x)y = 0$ then Wronskian $W(x) = W(y_1, y_2)$ is either identically 0 on I or never 0.

Proof : $W = y_1 y_2' - y_2 y_1'$

$W' = y_1' y_2' + y_1 y_2'' - y_2' y_1' - y_2 y_1'' = y_1 y_2'' - y_2 y_1''$

$= -P(x) W$

$\Rightarrow W(x) = W(x_0) \exp\left[-\int_{x_0}^{x} P(x)\,dx\right]$ for some $x_0 \in I$

If $W(x_0) = 0 \Rightarrow W(x) = 0 \quad \forall x \in I$

If $W(x_0) \neq 0 \Rightarrow W(x) \neq 0 \quad \forall x \in I$ since exp is never zero.

Theorem : If $y_1(x)$ and $y_2(x)$ are two sol^n of $L(y) = 0$ then they are LD iff their Wronskian is identically zero on I.

Proof : If y_1 and y_2 are LD $\Rightarrow y_1 = k y_2 \Rightarrow y_1' = k y_2'$

$\Rightarrow W(y_1, y_2) = 0$

If $W(y_1, y_2) = 0 \Rightarrow y_1 y_2' - y_2 y_1' = 0$, Choose x_0

~~if $y_1(x_0) \neq 0 \Rightarrow$~~

~~If $y_1(x_0) = 0$~~ If $y_1(x) = 0 \quad \forall x \in I$

then $1 y_1 + 0 y_2 = 0$ hence y_1, y_2 are LD.

If $y_1(x)$ is not identicaly 0 then $y_1(x_0) \neq 0$ for some x_0

$\Rightarrow$ Some nbd J in which $y_1(x) \neq 0$

$$\Rightarrow \frac{y_1 y_2' - y_2 y_1'}{y_1^2} = 0 \quad \text{on } J$$

$$\Rightarrow \frac{d}{dx}\left(y_2/y_1 \right) = 0 \quad \text{on } J$$

$$\Rightarrow y_2 = k y_1 \quad \text{on } J$$

$$\Rightarrow y_2' = k y_1' \quad \text{on } J$$

$\Rightarrow$ in Perticular $y_2(x_0) = k y_1(x_0)$ and $y_2'(x_0) = k y_1'(x_0)$

$\Rightarrow$ By uniqueness theorem $y_2(x) = y_1(x) \quad \forall x \in I$.

Theorem : If P, Q are cont. fn. on I. Let y_1 and y_2 be LI soln. of $L(y) = 0$ on I, then $c_1 y_1(x) + c_2 y_2(x)$ is general soln. of the DE on I.

Proof : Let y be any solution of $L(y) = 0$. For y to be equal to $c_1 y_1 + c_2 y_2$ we must show that they agree at some point ~~x_0~~ in I and also their derivatives.

$$c_1 y_1(x_0) + c_2 y_2(x_0) = y(x_0)$$

$$c_1 y_1'(x_0) + c_2 y_2'(x_0) = y'(x_0)$$

Since $W(y_1, y_2) \neq 0$ non trivial solutions c_1, c_2 exist.

Remark : c_1 and c_2 above are also unique.

1. A set of any two soln. of this DE is called a Basis $B = \{y_1, y_2\}$
   $S = \{c_1 y_1 + c_2 y_2 \mid c_1, c_2 \in R\}$ is called span of B
2. By above theorem S is also the set of all soln. of the DE
3. Solution space is ~~called~~ two dimensional.

Theorem : If y_g is a general soln of $L(y) = 0$, and y_p is some solution of $L(y) = R(x)$ then

$$y_g + y_p \text{ is a general soln of } L(y) = R(x)$$

Proof : y be any soln of $L(y) = R(x)$ $\quad y - y_p$ is a soln of $L(y) = 0$

then $y - y_p = y_g$

$$y = y_p + y_g$$

Example : $x^2y'' - 2xy' + 2y = 0$ I is any interval not containing origin

$y'' - 2y'/x^2 + 2y/x^2 = 0$

$p(x) = -2/x$, $Q(x) = 2/x^2$ are cont. on I

Check solutions $y_1 = x$, $y_2 = x^2$

$W(y_1, y_2) = y_1 y_2' - y_2 y_1' = x \cdot 2x - x^2 \cdot 1 = x^2 \neq 0$ if $x \neq 0$

Hence General solⁿ is $y = C_1 x + C_2 x^2$

A particular solⁿ s.t. $y_p = 3$, $y' = 5$ at $x = 1$

$$\left.\begin{aligned} C_1 + C_2 &= 3 \\ C_1 + 2C_2 &= 5 \end{aligned}\right\} \Rightarrow \begin{aligned} C_1 &= 1 \\ C_2 &= 2 \end{aligned}$$

$y_p = x + 2x^2$

Example : Find solⁿ of $y'' + 9y = 3$

① $y'' + 9y = 0$

$\cos 3x$, $\sin 3x$

$W = \cos(3x) \cdot 3(\cos 3x) + 3\sin(3x) \sin 3x = 3 \neq 0$

② $y'' + 9y = 3$ $y = A$ $y' = y'' = 0$

$9A = 3 \Rightarrow y = 1/3$ is solⁿ of the DE

③ $y_g = C_1 \cos(3x) + C_2 \sin(3x) + 1/3$.

Example : $y'' + 2y' + y = e^{-x}$

① $y'' + 2y' + y = 0$

$y_1 = e^{-x}$, $y_2 = x e^{-x}$

$W(y_1, y_2) = e^{-2x}(-x + 1) + x e^{-2x} = e^{-2x} \neq 0$

② $y = A x^2 e^{-x}$

$A e^{-x} \left(\cancel{x^2} + 2(\cancel{2x} - \cancel{x^2}) + (-)(\cancel{2x} - \cancel{x^2}) + (2 - \cancel{2x}) \right) = e^{-x}$

$\Rightarrow 2A = 1 \Rightarrow A = 1/2$

$y_p = \frac{1}{2} x^2 e^{-x}$

③ General solⁿ $y_g = e^{-x}(C_1 + C_2 x + \frac{1}{2} x^2)$

2.1 Second Ordered, homogeneous Linear Diff. Eq. with constant co-efficients

The DE equations of the form

$$a y'' + b y' + c y = 0$$

where a, b, c are real constants with $a \neq 0$

The sol^n can be obtained by substituting $y = e^{mx}$

$\Rightarrow \quad a m^2 + b m + c = 0$ Auxillary Equation, Characteristic eq.

- If $b^2 - 4ac \neq 0$ then the two distinct roots are

$$m_1 = -b/2a + \sqrt{b^2-4ac}/2a \,, \quad m_2 = -b/2a - \sqrt{b^2-4ac}/2a$$

then general solution $= C_1 e^{m_1 x} + C_2 e^{m_2 x}$

The two sol^n are LI since

$$W = \begin{vmatrix} e^{m_1 x} & e^{m_2 x} \\ m_1 e^{m_1 x} & m_2 e^{m_2 x} \end{vmatrix} = e^{(m_1+m_2)x} \begin{vmatrix} 1 & 1 \\ m_1 & m_2 \end{vmatrix}$$

$$= (m_2 - m_1)\, e^{(m_1+m_2)x} \neq 0 \quad \text{if } m_1 \neq m_2$$

- If $b^2 - 4ac = 0 \Rightarrow m_1 = m_2 = -b/2a$, $\quad y_1 = e^{m_1 x}$

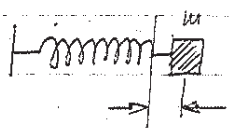
Try $y_2 = u(x) e^{m_1 x} \Rightarrow u(x) = (px + q) \qquad y_2 = x e^{m_2 x}$

$$W = \begin{vmatrix} e^{m_1 x} & x e^{m_2 x} \\ m_1 e^{m_1 x} & (m_2 x + 1) e^{m_2 x} \end{vmatrix} = e^{m_1 x} \neq 0$$

General Solution $y_g = (C_1 + C_2 x) e^{m_1 x}$.

Example : (Free Oscillations)

$$m \ddot{y} = -k y$$



$$\ddot{y} + \omega_0^2 y = 0 \qquad \omega_0^2 = k/m$$

two sol^n can be obtained by $p^2 + \omega_0^2 = 0 \Rightarrow p = \pm i \omega_0$

General Sol^n $\quad y_g = A e^{i\omega_0 t} + B e^{-i\omega_0 t}$

$$= C \sin(\omega_0 t) + D \cos(\omega_0 t)$$

$$= R \cos(\omega_0 t - \delta)$$

motion is always sinusoidal.

Example : (Damped Vibrations)

$$m \ddot{y} = -k y - c \dot{y} \qquad k, c > 0$$

$$\ddot{y} + 2\gamma \dot{y} + \omega_0^2 y = 0 \qquad 2\gamma = c/m \quad \omega_0^2 = k/m$$

Auxillary Eq. $\quad p^2 + 2\gamma p + \omega_0^2 = 0$

if $\quad b^2 - 4ac = 4\gamma^2 - 4\omega_0^2 < 0 \Rightarrow \gamma^2 < \omega_0^2$

$$p_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2} \qquad p_2 = -\gamma - \sqrt{\gamma^2 - \omega_0^2}$$

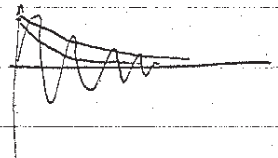
$$y_g(t) = e^{-\gamma t} \left(C_1 e^{i\omega t} + C_2 e^{-i\omega t} \right) \qquad \omega = \sqrt{\omega_0^2 - \gamma^2}$$

$$= A e^{-\gamma t} \cos(\omega t - \delta)$$

motion is sinusoidal, $\omega = \sqrt{\omega_0^2 - \gamma^2}$ and amplitude $A e^{-\gamma t}$.

if $\gamma = \omega_0$ then $p_1 = p_2 = -\gamma$

$$y_g = (A + Bt)e^{-\gamma t}$$



if $\gamma > \omega_0$ then $p_1 = -\gamma + \sqrt{\gamma^2 - \omega_0^2}$ $p_2 = -\gamma \pm \sqrt{\gamma^2-\omega_0^2}$

~~since~~ $\gamma > \omega_0$; $\gamma^2 > \gamma^2 - \omega_0^2 > 0 \Rightarrow \gamma > \sqrt{\gamma^2 - \omega_0^2} > 0$

then p_1 and $p_2 < 0$

$$y_g = Ae^{+p_1 t} + Be^{+p_2 t} \to 0 \text{ as } t \to \infty$$

2.2 Second-Ordered Nonhomogeneous Linear DE with const co-efficients

2.2.1 Method of variation of Parameters.

$$\mathcal{L}(y) = f(x) \quad : \quad ay'' + by' + cy = f(x)$$

General Soln. $y_g = y_p + (C_1 y_1 + C_2 y_2)$

Where y_1 and y_2 are sol^n of $\mathcal{L}(y) = 0$, y_p remains to be found.

Assume a solution of the form

$$u_1(x)\, y_1(x) + u_2(x)\, y_2(x)$$

$$u_1 \mathcal{L}(y_1) + u_2 \mathcal{L}(y_2) +$$
$$a(y_1 u_1'' + y_2 u_2'') + 2a(y_1' u_1' + y_2' u_2') + b(u_1' y_1 + u_2' y_2) = f(x) \quad - \text{①}$$

if
$$\left.\begin{aligned} y_1 u_1' + y_2 u_2' &= 0 \\ y_1' u_1' + y_2' u_2' &= f(x)/a \end{aligned}\right\} \text{ then ① is satisfied.}$$

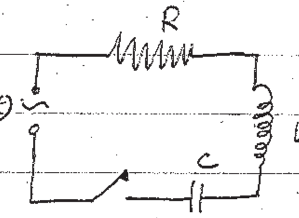
Let $F(x) = f(x)/a$

then $u_1' = -\dfrac{y_2 F}{W(y_1, y_2)}$ $\quad u_2' = \dfrac{y_1 F}{W(y_1, y_2)}$

$$\Rightarrow \quad y_p(x) = + \int_{x_0}^{x} \frac{\left(y_2(x)\, y_1(t) - y_1(t)\, y_2(x)\right) F(t)\, dt}{W(y_1, y_2)(t)}$$

Example : (Electrical Network)

$$E(t) = L \frac{dI}{dt} + RI + \frac{Q}{C}$$



$$E_0 e^{i\omega_0 t} = L \frac{d^2 Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C} Q$$

Assume $R^2 \neq 4LC$

A.E. $Lp^2 + Rp + 1/C = 0 \Rightarrow p_{2,1} = -R/2L \pm \sqrt{R^2/4L^2 - 1/LC} \Rightarrow y_{2,1} = e^{p_{2,1} t}$

$$u_1' = -\frac{e^{p_2 t}\, (E_0/L)\, e^{i\omega_0 t}}{W(y_1, y_2)} \qquad W(y_1, y_2) = (p_2 - p_1)\, e^{(p_1 + p_2)t}$$

$$= -\frac{E_0\, e^{i\omega_0 t - p_1 t}}{L(p_2 - p_1)}$$

$$\Rightarrow u_1 = -\frac{E_0\, e^{(i\omega_0 - p_1)t}}{L(p_2 - p_1)(i\omega_0 - p_1)} \qquad \text{similarly } u_2 = \frac{E_0\, e^{(i\omega_0 - p_2)t}}{L(p_2 - p_1)(i\omega_0 - p_2)}$$

Particular Sol^n then is

$$y_p = u_1(y_1) + u_2 y_2 = \frac{E_0 e^{i\omega_0 t}}{2(P_2-P_1)}\left(\frac{-1}{i\omega_0 - P_1} + \frac{1}{i\omega_0 - P_2}\right)$$

$$= \frac{E_0 e^{i\omega_0 t}}{i\omega_0\left(R + i\left(\omega_0 L - \frac{1}{\omega_0 C}\right)\right)} \qquad z - \text{impedance.}$$

$$q(t) = C_1 e^{-P_1 t} + C_2 e^{-P_2 t} + \frac{E_0}{i\omega_0 z} e^{i\omega_0 t}$$

both $e^{-P_1 t} \to 0$ and $e^{-P_2 t} \to 0$ as $t \to \infty$

$q_{steady}(t) = (E_0 / i\omega_0 z)\, e^{i\omega_0 t}$

$i_{steady}(t) = (E_0/z)\, e^{i\omega_0 t}$

2.2.2 Method of undetermined co-efficients

If RHS of SONHLDE is a function from a set of UDC functions; defined by

$\{ f \,/\, f$ either x^n $(n \geq 0;\text{int})$, $e^{\alpha x}$ $(\alpha \neq 0)$, $\sin(\beta x + \varphi)$, $\cos(\beta x + \varphi)$ $(\beta \neq 0)$, or product of these. $\}$

All these fn have finite no. of LI derivatives.

UDC Set of a function f is a set of all possible LI UDC fn that appear in the derivative of f.

Example : x^4, e^{2x}, x^3, $\sin(3x)$ UDC functions

$x^2 e^{-x}$, $x^3 \sin(3x)$, $e^{2x} \cos 4x$, UDC functions

$x^3 + x^2 e^x + \sin(x)$, UDC function.

Example : UDC set of x^4 is $\{x^4, x^3, x^2, x, 1\}$

" x^n is $\{x^n, x^{n-1}, \ldots, 1\}$

$e^{\alpha x}$ is $\{e^{\alpha x}\}$

$\sin(\beta x)$ is $\{\sin\beta x, \cos\beta x\}$

Example : UDC set of $x^3 e^{\alpha x}$ is $\{x^2 e^{\alpha x}, x^2 e^{\alpha x}, x e^{\alpha x}, e^{\alpha x}\}$

$x^2 \sin(\beta x)$ is $\{x^2, x, 1\} \times \{\sin\beta x, \cos\beta x\}$

To solve the diff. eq. $L(y) = f(x)$, where $f(x)$ is a UDC function. Let $\{y, u_1, u_2, \ldots, u_k\}$ be a UDC set of f

Assume solution to be $y_p = A_1 u_1 + A_2 u_2 + \cdots + A_k u_k$, $A_1, \ldots, A_k$ are const. It only remains to determine these. ~~Hence~~.

Example : (Forced vibrations without Damping)

$$y'' + \omega_0^2 y = F_0 \cos\omega t$$

UDC Fn in RHS $\cos\omega t$

UDC set is $\{\cos\omega t + \sin\omega t\}$

Assume a soln. $y = A\cos\omega t + B\sin\omega t$

~~$y'' = -A\omega^2 - B\omega^2 \sin$~~ $y'' = -A\omega^2\cos\omega t - B\omega^2\sin\omega t$

Substituting in the ~~soln~~ Eq.

$$+A(\omega_0^2-\omega^2)\cos\omega t + B(\omega_0^2-\omega^2)\sin\omega t = F\cos\omega t$$

$$\Rightarrow B = 0, \quad A = F/(\omega_0^2-\omega^2)$$

$$y_p = F\cos\omega t/(\omega_0^2-\omega^2)$$

$$y_g = F/(\omega_0^2-\omega^2)\cos(\omega t) + C\cos(\omega_0 t - \delta)$$ Resonance.

Remark ~~Example~~ If one of the fn in UDC set is already a ~~member~~ soln. of $\mathcal{L}(y)=0$ then ~~choo~~ multiply the whole set by a lowest integral power of x s.t. it contains no soln. to $\mathcal{L}(y)=0$

Example: $y''+2y' = 3+4\sin x$ Soln of $y''+2y'=0$ are $y_1 = 1$ $y_2 = e^{-2x}$

RHS contains two terms 1, $\sin x$

UDC set of 1 $\{1\} \longrightarrow \{x\}$

UDC set of $\sin x$ $\{\sin x, \cos x\}$

$$y_p = Ax + B\sin x + C\cos x$$

Substituting in eq.

$$-B\sin x - C\cos x + 2A + 2B\cos x - 2C\sin x = 3 + 4\sin x$$

$$\left.\begin{array}{l} 2A + 0B + 0C = 3 \\ 0 - B - 2C = 4 \\ 0 + 2B - C = 0 \end{array}\right\} \quad \begin{array}{l} A = \frac{3}{2} \\ B = -5/16 \\ C = -\cancel{5/8} \end{array}$$

$$y_p = \tfrac{3}{2}x - \tfrac{5}{16}(\sin x + 2\cos x)$$

$$y_g = C_1 + \tfrac{3}{2}x + C_2 e^{-2x} - 5/16(\sin x + 2\cos x)$$

2.3 Second Ordered DE (Linear)

Definition. DE of the form

$$\mathcal{L}(y) = y'' + P(x)y' + Q(x)y = R(x)$$

P, Q, R cont real fn on I.

Remark: homogeneous eq. if $R(x) = 0 \ \forall x \in I$

Non-homogeneous if $R(x) \neq 0$ for some $x \in I$

2.3.1 Reduction of order by known solution for Homogeneous DE

Consider $\mathcal{L}(y) = 0$. Let y_1 be sol^n. Assume $y_2 = u y_1$

$$\mathcal{L}(y_2) = u\mathcal{L}(y_1) + u'' y_1 + u'(2y_1' + p y_1) = 0$$

Reduces to $\quad u'' y_1 + u'(2y_1' + py_1) = 0$

then $\quad u' = k_1/y_1^2 \exp(-\int p\,dx)$

and $\quad u = k_1 \int (1/y_1^2) \exp(-\int p\,dx)\,dx + k_2$

and $\quad y_2 = u y_1$.

Example: (Legendre Equation)

$$(1-x^2) - 2xy' + 2y = 0 \qquad I = (0,1)$$

Verify one sol^n $y_1(x) = x$, $y_2 = xu$

$$\cancel{\text{Also}}\ -\int p(x)dx = -\ln(1-x^2) \qquad p(x) = -2x/(1-x^2)$$

$$Q(x) = 2/(1-x^2)$$

$$u' = (1/x^2)\exp(-\ln(1-x^2))$$

$$= (1/x^2)(1/1-x^2)$$

$$= \frac{1}{x^2} + \frac{1}{1+x} + \frac{1}{1-x}$$

$$u = -1/x + \ln\left(\frac{1+x}{1-x}\right)$$

$$y_2 = x\ln\left(\frac{1+x}{1-x}\right) - 1.$$

2.3.2 Series solutions for homogeneous DE

If the co-efficients functions $P(x)$ and $Q(x)$ are analytic functions of x, then the sol^n is also analytic at x_0.

Analytic $\quad f(x) = \sum a_n (x-x_0)^n \qquad |x-x_0| < r_0$

$$f'(x) = \sum n a_n (x-x_0)^{n-1}$$

Hence formally can be obtained.

Example : $y'' + \omega_0^2 y = 0$

Assume $y = \sum_0^\infty a_k x^k = a_0 + a_1 x + a_2 x^2 + \cdots$

$y' = \sum_1^\infty a_k k x^{k-1} = a_1 + 2a_2 x + 3a_3 x^2 + \cdots$

$y'' = \sum_2^\infty a_k (k-1) k x^{k-2} = 2a_2 + 6a_3 x + 12a_4 x^2$

Change index $k' = k-2 \Rightarrow k = k'+2 \qquad k=2 \Rightarrow k'=0$

$= \sum_0^\infty a_{k+2} (k+1)(k+2) x^k$

Substituting in DE

$$\sum_0^\infty \left[a_{k+2} (k+1)(k+2) + \omega_0^2 a_k \right] x^k = 0$$

$$\Rightarrow \qquad a_{k+2} = -\frac{\omega_0^2}{(k+1)(k+2)} a_k \qquad \forall k.$$

$\Rightarrow$ Choose a_0 $\quad a_2, a_4 \cdots$ determined

$\Rightarrow$ Choose a_1 $\quad a_3, a_4 \cdots$ determined

$\left.\right\}$ two arbitrary const.

(a) $a_0 = 1$ and $a_1 = 0$

$a_2 = -\omega_0^2/2!$, $a_4 = \omega_0^4/4!$, $\cdots\cdots$, $a_{2n} = \omega_0^{2n}/(2n)!\,(-1)^n$

$y_1 = 1 - \frac{1}{2!}(\omega_0 x)^2 + \frac{1}{4!}(\omega_0 x)^4 \cancel{+} \cdots\cdots$

$= \cos(\omega_0 x)$

(b) $a_0 = 0$ and $a_1 = 1$

$a_3 = -\omega_0^3/3!$, $a_5 = \omega_0^5/5!$, $\cdots$

$y_2 = x - 1/3!\,(\omega_0 x)^3 + \cdots$

$= \sin(\omega_0 x)$.

(c) $P(x) = 0 \quad Q(x) = \omega_0^2$ are analytic on $\mathbb{R}$ so is y_1 and y_2

(d) $y_g = a_0 \cos(\omega_0 x) + a_1 \sin(\omega_0 x)$

Example : $y'' - xy = 0$

$P(x) = 0$, $Q(x) = -x$.

$y = \sum_0^\infty a_k x^k = a_0 + a_1 x + \cdots$

$xy = \sum_1^\infty a_{k-1} x^k = a_0 x + a_1 x^2 + \cdots$

$y'' = \sum_2^\infty a_k k(k-1) x^{k-2} = 2a_2 + 6a_3 x + \cdots$

$= \sum_0^\infty a_{k+2} (k+1)(k+2) x^k$

Substituting in DE

$$2a_2 + \sum_{k=1}^\infty \left[a_{k+2} (k+1)(k+2) - a_{k-1} \right] x^k = 0$$

$\Rightarrow a_2 = 0$ and $a_{k+2} = \frac{a_{k-1}}{(k+1)(k+2)}$

(a) $a_0 = 1$ and $a_1 = 0$

$a_3 = 1/3.2$, $a_6 = 1/6.5.3.2$

$a_1 = a_2 = a_4 = a_5 \cdots = 0$

$$y_1 = 1 + \frac{x^3}{3.2} + \frac{x^6}{6.5.3.2}$$

(b) $a_0 = 0$ and $a_1 = 1$

$a_0 = a_2 = a_3 = a_5 = a_6 = \cdots = 0$

$a_4 = 1/4.3$, $a_7 = 1/7.6.4.3$, $\cdots$

$$y_2 = x + \frac{x^4}{4.3} + \frac{x^7}{7.6.4.3} + \cdots$$

(c) General solⁿ $y_g = C_1 y_1 + C_2 y_2$

(d) Convergence of the solⁿ

Ratio $\left|\frac{x^3}{(k+2)(k+1)}\right| \to 0$ as $k \to \infty$ $\frac{a_{k+2}}{a_{k-1}} x^3 = \frac{x^3}{(k+2)(k+1)}$

(e) $y_1(0) = 1$ $y_1'(0) = 0$

$y_2(0) = 0$ $y_2'(0) = 1$

$$W(y_1, y_2)(0) = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1 \neq 0$$ hence nonzero always.

2.3.3 Legendre Equation.

Origin : Consider Laplace equation Electrostatics.

$\nabla^2 V = 0$. if $V(r,\theta,\phi) = R(r)\,\Sigma(\theta)$ independent of ϕ

then $$\frac{1}{R}\frac{\partial}{\partial r}\left(r^2\frac{\partial R}{\partial r}\right) = -\frac{1}{\Sigma\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial\Sigma}{\partial\theta} = \ell(\ell+1)$$

$\Rightarrow$ $\sin\theta\, \partial^2\Sigma/\partial\theta^2 + \cos\theta\, \partial\Sigma/\partial\theta + (\ell+1)\ell\sin\theta\,\Sigma = 0$

$\Rightarrow$ put $\cos\theta = x$ $\Rightarrow$ $d/d\theta = -\sin\theta\, d/dx$

$\Rightarrow$ $(1-x^2)\, d^2\Sigma/dx^2 - 2x\, d\Sigma/dx + (\ell+1)\ell\,\Sigma = 0$

Interval : $P(x) = -2x/(1-x^2)$, $Q(x) = \ell(\ell+1)/(1-x^2) = \ell(\ell+1)\sum_k x^{2k}$

Radius of convergence $= 1$ for both series.

Solutions : Choose $y = \sum_k C_k x^k$.

$y' = \sum_1 k C_k x^{k-1}$

$-2x y' = -2\sum_1 k C_k x^k$

$y'' = \sum_2 k(k-1) C_k x^{k-2} = \sum_0 (k+1)(k+2) C_{k+2} x^k$

$-x^2 y'' = \sum_2 k(k-1) C_k x^k$

$$\sum_0 (k+2)(k+1) C_{k+2} x^k - \sum_2 (k)(k-1) C_k x^k - 2 \sum_1 k C_k x^k$$
$$+ (l)(l+1) \sum_0 C_k x^k.$$

$$\Rightarrow 2C_2 + 6C_3 x - 2C_1 x + l(l+1)(C_0 + C_1 x)$$
$$+ \sum_{k=2}^{\infty} \left((k+1)(k+2) C_{k+2} - k(k-1)C_k - 2kC_k + (l)(l+1) C_k \right) x^k = 0$$

$$\Rightarrow C_2 = \frac{-l(l+1)}{2} C_0$$

$$\Rightarrow C_3 = \frac{2 - (l)(l+1)}{6} C_1$$

$$\Rightarrow C_{k+2} = \frac{k(k+1) - l(l+1)}{(k+1)(k+2)} C_k$$
$$= -\frac{(l-k)(l+k+1)}{(k+1)(k+2)} C_k$$

$$\Rightarrow C_{2m} = \frac{(l+2m-1)(l+2m-3)\cdots(l+1)(l-2m+2)(l-2)(l)}{(2m)!} C_0$$

$$\Rightarrow C_{2m+1} = \frac{(l+2m)\cdots(l+2)(l-1)\cdots(l-2m+1)}{(2m+1)!}$$

$$y = \sum_{k=0}^{\infty} C_k x^k$$

Choose $y_1(0) = 1$ and $y_1'(0) = 0 \Rightarrow C_0 = 1, C_1 = 0$

$\Rightarrow y_1 =$ Series of even powers

Choose $y_2(0) = 0$ and $y_2'(0) = 1$

$\Rightarrow y_1 =$ Series of odd powers

General soln $y_g = c_1 y_1 + c_2 y_2$

Polynomials If $l = 2m$ for some integer m then

ϕ_1 is a polynomial of degree $2m$, ϕ_2 is not

If $l = 2m+1$ for some integer m then

ϕ_2 is a polynomial of degree $2m+1$, ϕ_1 is not.

These polynomials are called Legendre polynomials and dented by P_n

$P_0 = 1$ $\qquad$ $P_1 = x$

$P_2 = 1 - 3x^2$ $\qquad$ $P_3 = x - 5/3\, x^3$

$P_4 = 1 - 10x^2 + 35/3\, x^4$ $\qquad$ $P_5 = x - 14/3\, x^3 + 21/5\, x^5$

If $P_l(1) = 1 \quad \forall\, l.$ $\qquad$ $P_1 = x$

$P_0 = 1$ $\qquad$ $P_3 = 3/2 \left(5/3\, x^3 - x \right)$

$P_2 = + \frac{1}{2} (3x^2 - 1)$ $\qquad$ $P_5 = 15/8 \left(21/5\, x^5 - 14/3\, x^3 + x \right)$

$P_4 = 3/8 \left(35/3\, x^4 - 10x^2 + 1 \right)$

Rodrigue's Formula

$$P_\ell(x) = \frac{1}{2^\ell \ell!} \frac{d^\ell}{dx^\ell}(x^2-1)^\ell$$

Show by induction that if $u(x) = (x^2-1)^\ell$

$$(x^2-1)u^{(k)} = 2x(\ell-(k-1))u^{(k-1)} + 2\sum_{i=0}^{k-2}(n-i) \cdot u^{(k-2)}$$

and prove that $u^{(\ell)}$ satisfies Legendre equation with ℓ

$P_\ell(x)$ as defined by Rodrigues formula is a polynomial with degree ℓ. Must be the polynomial sol^n of Legendre Eq.

Generating Function for Legendre Polynomials

Let $\phi(x,h) = (1-2xh+h^2)^{-1/2}$ $\qquad |h|<1$

$$= \sum_0^\infty P_k(x) \cdot h$$

Show by explicit expansion. Proof in tutorial.

Orthogonality If P and Q are two polynomials on $I=[-1,1]$ then P,Q are orthogonal to each other if

$$\int P(x)Q(x)\,dx = 0$$

- Legendre Polynomials are orthogonal to each other.
- P,Q are LI if they are orthogonal.

Recurrence Relations:

2.3.4 LE with regular singular Points

$$a(x)y'' + b_1(x)y' + a_2 y = 0$$

If $a(x_0) = 0$ then x_0 is called singular point of the equation

Existence theorem does not apply.

If Equation can be written in the form

$$(x-x_0)^2 y'' + b_1(x)(x-x_0)y' + b_2(x)\,y = 0$$

then x_0 is a reg. sing point if b_1 and b_2 are analytic at x_0.

By applying a modified series method may obtain solⁿ for these problems.

Example: ① $x^2y'' + (x+x^2)y' - y = 0$

$x=0$, regular

② $xy'' + 4y = 0$ $\quad x=0 \Rightarrow$ not regular (why?)

③ $(1-x^2)y'' - 2xy' + 2y = 0$

$x = \pm 1$ regular at both points

④ $x^2y'' + \sin x\, y' + \cos x\, y = 0$

$x = 0$ Regular.

2.3.5 The Bessel Equation.

$$x^2y'' + xy' + (x^2 - p^2)y = 0$$

$b_1(x) = 1$ and $b_2(x) = x^2 - p^2$ are analytic at $x=0$.

Write as $x(xy')' + (x^2-p^2)y = 0$

Substitute $y = x^r \sum c_k x^k$

$$y' = \sum_k (k+r)c_k x^{k+r-1}$$

$$(xy') = \sum_k (k+r)c_k x^{k+r}$$

$$(xy')' = \sum_k (k+r)^2 c_k x^{k+r-1}$$

$$x(xy')' = \sum_k (k+r)^2 c_k x^{k+r}$$

$$\sum_k [(k+r)^2 - p^2]c_k x^{k+r} + \sum_k c_k x^{k+r+2}$$

Co-eff: of x^r $\quad c_0(r^2 - p^2) = 0$ then indicial Equation

$r = \pm p.$ $\quad$ if $c_0 \neq 0$

x^{r+1} $\quad ((r+1)^2 - p^2)c_1 = 0 \Rightarrow c_1 = 0$

In general $[(k+r)^2 - p^2]c_k = -c_{k-2}$

$$\Rightarrow c_k = -\frac{c_{k-2}}{(k+r)^2 - p^2}$$

If you choose $r = p$

then $$a_k = -\frac{a_{k-2}}{(k+p)^2 - p^2} = -\frac{a_{k-2}}{k(k+2p)}$$

$$a_{2k} = -\frac{a_{2k-2}}{2^2 k(k+p)}$$ since all odd terms are zero.

Let $$a_2 = -\frac{a_0}{2^2(1+p)} = -\frac{a_0\,\Gamma(1+p)}{2^2\,\Gamma(2+p)}$$

$$a_4 = -\frac{a_2}{2^3(2+p)} = \frac{a_0}{2 \cdot 2^4 (2+p)(1+p)} = \frac{a_0\,\Gamma(1+p)}{2\,2^4\,\Gamma(3+p)}$$

$$a_6 = -\frac{a_4}{2^2 \cdot 3 \cdot (3+p)} = \frac{-a_0\,\Gamma(1+p)}{2^6\,3!\,\Gamma(4+p)}$$

$$a_{2k} = \frac{(-1)^k\,\Gamma(1+p)}{2^{2k}\,k!\,\Gamma(k+p+1)}\,a_0$$

$$J_p(x) = y = a_0 x^p\,\Gamma(1+p)\left[\sum_k \frac{(-1)^k}{k!\,\Gamma(k+p+1)}\left(\frac{x}{2}\right)^{2k}\right]$$

First kind. Bessel functions

Second Sol^n is given by

$$J_{-p}(x) = a_0 x^{-p}\,\Gamma(1-p)\left[\sum_k \frac{(-1)^k}{k!\,\Gamma(k-p+1)}\left(\frac{x}{2}\right)^{2k}\right.$$

2.4 Non Homogeneous Dift. Eq. (Method of variation of Parameters.

$\mathcal{L}(y) = y'' + P(x)\,y' + Q(x)\,y = R(x)$

P, Q, R cont. on I.

$\mathcal{L}(y) = 0$ Ass. homo. Equation

Let y_1 and y_2 be LI soln of homogeneous equation

Assume $y = u_1 y_1 + u_2 y_2$ u_1, u_2 fn of x.

Substitution yields

$$\mathcal{L}(y) = \mathcal{L}(y_1) + \mathcal{L}(y_2) + (C_1'' y_1 + C_2'' y_2) + 2(C_1' y' + C_2' y_2') + P(x)(C_1' y_1 + C_2' y_2)$$
$$= R(x)$$

then
$$\left.\begin{aligned} C_1' y_1 + C_2' y_2 &= 0 \\ C_1' y_1' + C_2' y_2' &= R(x) \end{aligned}\right\} \text{ satisfies DE.}$$

$$\Rightarrow \quad C_1 = -\int \frac{y_2(x)\,R(x)}{W(y_1, y_2)(x)}\,dx$$

Mixup in C and u.

$$\Rightarrow \quad C_2 = +\int \frac{y_1(x)\,R(x)}{W(y_1, y_2)(x)}\,dx$$

Example : $x^2 y'' - x y' + y = x^2$ $x > 0$

$P(x) = -1/x$ $R(x) = 1$

$y_1 = x$ $y_2 = x \ln x$

$W(y_1, y_2) = x$

$u_1 = -\int x \ln x \cdot 1/x \; dx = -x \ln x + x$

$u_2 = +\int x/x \; dx = x$

$y = x^2$.

Chapter 3.1, Laplace Equation.

$$\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$$

~~$\nabla^2 u = 0$ Electr~~

$$\nabla^2 u = 0$$

u: Electrostatic Pot.
: Gravitational Pot.
: A comp. of Vector potential.

Motivated by several problems in physics.

3.1.1 Seperation of Variables, Cartesian - coordinates.

$$\nabla^2 u = \left(\partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2\right) u = 0$$

Clearly, if one can assume (by whatever means)

$$u = u_x(x)\, u_y(y)\, u_z(z)$$

$$\Rightarrow \frac{1}{u_x}\frac{\partial^2 u_x}{\partial x^2} + \frac{1}{u_y}\frac{\partial^2 u_y}{\partial y^2} + \frac{1}{u_z}\frac{\partial^2 u_z}{\partial z^2} = 0$$

$$\Rightarrow \frac{\partial^2 u_x}{\partial x^2} + l^2 u_x = 0\ ; \quad \frac{\partial^2 u_y}{\partial y^2} + m^2 u_y = 0\ ; \quad \frac{\partial^2 u_z}{\partial z^2} + n^2 u_z = 0$$

where $l^2 + m^2 + n^2 = 0$. (*)

Sol^n for each partial DE are known

$$u_x^l(x) = A_x^l e^{ilx} + B_x^l e^{-ilx}$$

$$u_y^m(y) = A_y^m e^{imy} + B_y^m e^{-imy}$$

$$u_z^n(z) = A_z^n e^{inz} + B_z^n e^{-inz}$$

Though l, m, n ~~are~~ is any triplet satisfying (*)

hence $u(x,y,z) = \sum_{l,m} A_{lm}\, u_x^l(x)\, u_y^m(y)\, u_z^n(z)$.

Example, Find ES potential inside a box (rectangular with sides a, b, c). s.t.

$u = 0 \quad \forall \quad x=0,\ x=a,\ y=0,\ y=b,\ z=0$, and $u = u_0$ for $z = c$

since $u(0, y, z) = 0 \Rightarrow u_x^l(0) = 0$ and $u_x^l(a) = 0$ for each l.

$$u_x^l(0) = A_x^l + B_x^l = 0$$

$$u_x^l(a) = 2iA_x^l \sin(la) = 0 \Rightarrow la = r_l \pi \Rightarrow l = r_l\pi/a$$

$$\Rightarrow u_x^l(x) \sim \cancel{\sin h(\tfrac{r_l \pi}{a} x)} = \sin(r_l \pi x/a)$$

$\parallel^{ly}$ $\quad u_y^m(y) \sim \sin(r_m \pi y/b) \qquad$ ~~then~~ $mb = r_m\pi$

then $\quad u_z^n(z) = 0 \Rightarrow u_z^n(z) \sim \sin(nz)$

$$n^2 = -(l^2 + m^2) = -\pi^2\left(r_l^2/a^2 + r_m^2/b^2\right) < 0$$

$\Rightarrow n$ is complex

$$\Rightarrow u_z(z) \sim A_z \sinh(nz) \qquad u_z(c) = V_0$$

$$\Rightarrow A = V_0/\sinh(nc)$$

Use fourier series.

$$u(x,y,z) = \sum_{l,m} \frac{V_0}{\sinh(nc)} \sin\left(\frac{r_l \pi x}{a}\right) \sin\left(\frac{r_m \pi y}{b}\right) \sinh(nz)$$

3.1.2 Seperation of Variables in Spherical Co-ordinates

$$\nabla^2 u = -k^2 u \quad \boxed{\text{Helmholtz eq.}}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2} = 0$$

$$u = R(r)\,T(\theta)\,P(\phi)$$

$$\therefore \frac{d^2 p}{\partial\phi^2} + m^2 p = 0$$

$$\therefore \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial T}{\partial\theta}\right) - \frac{m^2}{\sin^2\theta}T + \ell(\ell+1)T = 0$$

$$\therefore \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) - \frac{(\ell)(\ell+1)}{r^2}R + k^2R = 0$$

ⓐ $p \sim e^{\pm im\phi}$, $\{\sin m\phi, \cos m\phi\}$

ⓑ $$(1-x^2)\frac{d^2T}{dx^2} - 2x\frac{dT}{dx} + \left(\ell(\ell+1) - \frac{m^2}{(1-x^2)}\right)T = 0 \Big\} \text{ put } x = \cos\theta$$

Associated Legendre Equation $\quad m=0 \Rightarrow$ Legendre Equation

$$P_\ell^m(x) = (1-x^2)^{m/2}\frac{d^m}{dx^m}P_\ell(x), \quad P_\ell^0(x) = P_\ell(x)$$ other solⁿ is infinite

ⓒ $$r^2\frac{d^2R}{dr^2} + 2r\frac{dR}{dr} + \left(\ell(\ell+1) - k^2r^2\right) = 0$$

if $k=0$ $\quad r^\ell,\ r^{-\ell-1}$

if $k\neq 0$ $\quad j_\ell(kr),\ n_\ell(kr)$

Example: A uniform sphere is kept in an uniform electric field $\vec{E}$ find surface charge density.

$\nabla^2 u = 0$ if u is independent of $\phi \Rightarrow m=0$

$$u = \sum_\ell \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right)P_\ell(\cos\theta)$$

$u(r=R) = 0 \quad\Rightarrow\quad B_\ell = -A_\ell R^{2\ell+1}$

$u(r\to\infty) = -E_0 r\cos\theta \quad\Rightarrow\quad A_1 = -E_0$ and $A_\ell = 0$

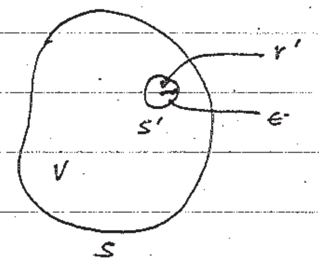
$$u(r,\theta) = -E_0\left(r - \frac{R^3}{r^2}\right)\cos\theta$$

$$\sigma = \epsilon E_n = -\epsilon_0\frac{\partial u}{\partial r}\Big|_R = +\frac{3\epsilon_0 R^3}{R^3}\cos\theta = 2\epsilon_0\cos\theta$$

3.2 Method of Green's functions.

To solve dirichlet problem inside a volume V, described by

$$\nabla^2 u = 0 \quad \text{and} \quad u(s) \text{ is given}$$



Let $\psi = 1/|\vec{r}-\vec{r}'|$, draw a sphere of radius ϵ about $\vec{r}'$.

Let $\vec{J} = u\nabla\psi - \psi\nabla u$

then $\vec{\nabla}\cdot\vec{J} = 0 = (u\nabla^2\psi - \psi\nabla^2 u) + (\nabla u\cdot\nabla\psi - \nabla\psi\cdot\nabla u)$

by divergence theorem

$$0 = \int_V \nabla\cdot J\, dv = \int_{S+S'} J\cdot d\vec{s}$$

$$d\vec{s} = \hat{n}\,\epsilon^2 d\Omega$$

Then $u(r) = u(r') + \epsilon\,\hat{n}\cdot\nabla u(r') + O(\epsilon^2)$ $\qquad \psi = \frac{1}{\epsilon} \quad \nabla\psi\cdot\hat{n} = \frac{1}{\epsilon^2}$

$$\int_{S'} J\cdot d\vec{s} = \int \left(u(r') + O(\epsilon)\right)\cdot\frac{1}{\epsilon^2}\cdot\epsilon^2\, d\Omega - \int \frac{1}{\epsilon}\nabla u\cdot\hat{n}\,\epsilon^2\, d\Omega$$

$$= 4\pi u(r') + O(\epsilon)$$

Then $\displaystyle\lim_{\epsilon\to 0}\int_{S'} J\cdot d\vec{s} = 4\pi u(r')$

$$\Rightarrow u(r') = \frac{1}{4\pi}\left[\int_S \left(u\,\vec{\nabla}\psi\cdot d\vec{s} - \psi\,\vec{\nabla}u\cdot d\vec{s}\right)\right] \qquad \nabla u \text{ on } S \text{ is needed}(!)$$

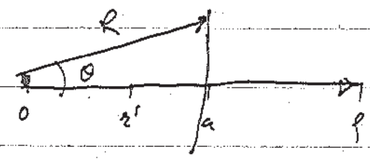
Find $\displaystyle G(r,r') = \frac{1}{|r-r'|} + H(r,r')$

s.t. $\nabla^2 H(r,r') = 0$

and $G(r,r') = 0$ on S

$$\Rightarrow u(r') = \frac{1}{4\pi}\int_S u\,\vec{\nabla}_{\vec{r}}\left(G(r,r')\right)\cdot d\vec{s}$$

Example. To solve $\nabla^2 u = 0$ inside a sphere of radius a, where $u(a,\theta,\phi)$ is specified.



1. Fix r' choose $\vec{\rho} = a^2/r'^2\ \vec{r}'$

Choose $\displaystyle H(r,r') = -\frac{a}{r'|\vec{r}-\vec{\rho}|}$

$\nabla^2 H = 0$ since $|\vec{\rho}| > a$

2. $\displaystyle G = \frac{1}{|r-r'|} - \frac{a}{r'|r-\vec{\rho}|}$

$|R-\vec{\rho}| = a/r'\ |R-r'|$ then $G(\bar{R},\vec{r}') = 0$ on S.

3. $\vec{\nabla} \vec{G}(r, r') \cdot \hat{n} \Big|_{\vec{R}} = \left(\frac{a^2 - r'^2}{a} \right) \frac{1}{|R - r'|^3}$

4. $u(r', \theta, \phi) = \frac{1}{4\pi} \int u(a, \theta, \phi) \cdot \left(\frac{a^2 - r'^2}{a} \right) \frac{1}{|R - r'|^3} a^2 \, d\Omega$

$$= \frac{a(a^2 - r'^2)}{4\pi} \int \frac{u(a, \theta, \phi) \sin\theta d\theta d\phi}{(a^2 + r'^2 - 2r'a\cos\theta)}$$

Remark : Solution to poisson Equation

$\nabla^2 u = \rho/\epsilon_0$ $\quad \rho(r)$ is known function

Let $G(r, r') = (|r - r'|)^{-1}$

$\nabla^2 G = + 4\pi \delta(r' - r)$

$u(r) = +\frac{1}{4\pi\epsilon_0} \int G(r, r') \rho(r') dr'$

$$\nabla^2 u = + \frac{1}{4\pi\epsilon_0} \int \nabla^2 G(r, r') \rho(r') dr'$$

$$= \frac{1}{\epsilon_0} \rho(r)$$

$$\Rightarrow u(r) = + \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r') dr'}{|r' - r|}$$

4\. Wave Equation

4.1 Introduction

- Vibrations of a string
- " of a membrane
- Sound waves
- Light waves

4.2 Elementary solution

Consider $$\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2} \qquad -\infty < x < \infty$$

Any function of the form
$$y = f(x+ct) + g(x-ct)$$
are solutions of the above equation.

If at $t=0$
$$y = \eta(x) \qquad \partial y/\partial t = v(x)$$

then
$$\left.\begin{aligned} f(x) + g(x) &= \eta(x) \\ cf'(x) - cg'(x) &= v(x) \\ f(x) - g(x) &= \frac{1}{c}\int_b^x v(\xi)\,d\xi \\ f(x) &= \frac{1}{2}\eta(x) + \frac{1}{2c}\int_b^x v(\xi)\,d\xi \\ g(x) &= \tfrac{1}{2}\eta(x) - \frac{1}{2c}\int_b^x v(\xi)\,d\xi \end{aligned}\right\} t=0$$

$$y = \frac{1}{2}\{\eta(x+ct) + \eta(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} v(\xi)\,d\xi$$

if $v(\xi)=0 \ \forall \xi$ at then
$$y = \frac{1}{2}\{\eta(x+ct) + \eta(x-ct)\} \qquad \text{d'Alembert's solution.}$$

If the boundary conditions are
$$\text{at } t=0 \quad y(x) = \eta(x) \quad \partial y/\partial t = v(x) \quad x \geq 0$$
$$\text{at } x=0 \quad y(x) = 0 \quad \forall t \geq 0.$$

Let
$$\begin{aligned} Y(x) &= \eta(x) & x \geq 0 \\ &= -\eta(-x) & x < 0 \\ V(x) &= v(x) & x \geq 0 \\ &= -v(-x) & x < 0 \end{aligned}$$

$$y = \frac{1}{2}\{Y(x+ct) + Y(x-ct)\} + \frac{1}{2c}\int_{x-ct}^{x+ct} V(\xi)\,d\xi$$

Satisfies the boundary conditions.
$$y = \begin{cases} \tfrac{1}{2}[\eta(x+ct) + \eta(x-ct)] & x \geq ct \\ \tfrac{1}{2}[\eta(x+ct) - \eta(ct-x)] & x < ct \end{cases}$$

4.2 Three dimensional waves

The equation $\nabla^2 \psi = \frac{1}{c^2} \frac{\partial^2 \psi}{\partial t^2}$

has sol^{n} of the form

$$\exp\{\pm i(lx + my + nz + kct)\}$$

if $k^2 = l^2 + m^2 + n^2$

then in Spherical co-ordinates

$$\psi(r,\theta,\phi) = (kr)^{-1/2}\left(A\, J_{n+1/2}(kr) + B\, J_{-n-1/2}(kr)\right) P_n^m(\cos\theta)\, e^{\pm im\phi \pm ikct}$$

m and n must be integers.

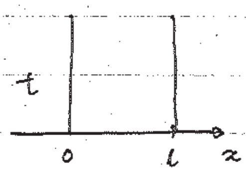
put $m = n = 0$

$$\psi(r) = \frac{1}{r} e^{\pm ikr \pm ikct} \qquad \text{Spherical wave}$$

$$\psi(r,\theta) = \frac{1}{r}\left[\frac{\sin(kr)}{kr} - \cos kr\right]\cos\theta\, e^{ikct} \qquad p \text{ wave} \qquad (n = 1,\ m = 0).$$

4.3 Vibrating strings: Method of seperation of variables

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2 y}{\partial t^2}$$



on a finite string of l and initial conditions $y(x,0) = \eta(x)$

put $y = X(x)T(t)$

$$X'' + k^2 X = 0 \Rightarrow \sin kx,\ \cos kx$$

$$\ddot{T} + k^2 v^2 T = 0 \Rightarrow \sin k v t,\ \cos \omega t \qquad \boxed{\omega = kv}$$

Then $y = \cancel{A\sin(kx+\omega t) +}$

$= A \sin kx \sin\omega t + B \sin kx \cos\omega t + C\cos kx \sin\omega t + D\cos kx \cos\omega t$

bc $\quad y(x,0) = \eta(x) \qquad y(0,t) = y(l,t) = 0 \qquad \dot{y}(x,0) = 0$

ⓐ $y(0,t) = 0 \Rightarrow C = D = 0$

ⓑ $y(l,t) = 0 \Rightarrow \sin kl\,(A\sin\omega t + B\cos\omega t) = 0$

$\Rightarrow kl = n\pi \Rightarrow k = n\pi/l \qquad$ n is an integer

$y = \cancel{A} \sin\left(\frac{n\pi}{l}x\right)\left[A\cos\omega t + B\sin\omega t\right] \qquad \omega = \frac{n\pi v}{l} \qquad$ n is an integer

ⓒ $\dot{y}(x,0) = 0 \Rightarrow B = 0$

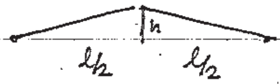
$$y = \sum_n A_n \sin\left(\frac{n\pi}{l}x\right)\cos\left(\frac{n\pi v}{l}t\right)$$

ⓓ $y(x,0) = \eta(x) = \sum_n A_n \sin\left(\frac{n\pi x}{l}\right)$

$$\Rightarrow A_n = \frac{2}{l}\int_0^l \eta(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

Example $\eta(x) = \frac{2h}{l} x \qquad x < l/2$

$$= \frac{2h}{l}(l - x) \qquad x \geq l/2$$



$$b_n = \frac{2}{l} \int \eta(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{8h}{\pi^2} \frac{(-1)^n}{n^2} \qquad n \text{ odd}$$

$$= 0 \qquad n \text{ even}$$

$$y = \frac{8h}{\pi^2}\left[\sin\left(\frac{\pi x}{l}\right)\sin\left(\frac{\pi v t}{l}\right) - \frac{1}{9}\sin\left(\frac{3\pi x}{l}\right)\sin\left(\frac{3\pi v t}{l}\right) + \dots\right]$$

5.0 Integral Transforms

5.1 Fourier ~~Transforms~~ Series

If a function f is integrable on a closed interval $[-\pi,\pi]$ the formal series

$$\sum_{n=-\infty}^{\infty} c_n e^{int}$$

with $$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\, e^{-int}\, dt$$

is called fourier series of f.

* The question of convergence

Theorem: If $\int_{-\pi}^{\pi} |f(t)|^2 dt$ exists, then fourier series converges in mean square sense

$$\lim_{N\to\infty} \int_{-\pi}^{\pi} \left| f(t) - \sum_{-N}^{N} c_n e^{int} \right|^2 dt = 0$$

- No proof, Weak convergence compared to uniform convergence

Parseval Identity

$$\int_{-\pi}^{\pi} |f(t)|^2 dt = 2\pi \sum_{-\infty}^{\infty} |c_n|^2$$

Example: $f(x) = -1 \quad -\pi \le x < 0$

$= 1 \quad 0 < x \le \pi$

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\, e^{-inx}\, dx$$

$$= \frac{1}{2\pi}\left[-\int_{-\pi}^{0} e^{-inx} dx + \int_{0}^{\pi} e^{-inx} dx \right]$$

if $n=0$, $c_n = 0$

$$= \frac{1}{2\pi}\left[-\frac{1-e^{in\pi}}{-in} + \frac{e^{-in\pi}-1}{-in} \right]$$

$$= -\frac{1}{2\pi}\left[\frac{-2+2(-1)^n}{in} \right] = +\frac{i}{\pi}\,\frac{(-1)^n - 1}{n}$$

$$f(x) = \frac{i}{\pi}\sum_{\neq 0, -\infty}^{\infty} \frac{(-1)^n-1}{n} e^{inx}$$

$$= -\frac{2}{\pi}\sum_{-\infty}^{\infty} \frac{(-1)^n-1}{n} \sin(nx)$$

Notice f is not continuos at 0. The RHS of above eq.

$$= \frac{f_+(0) + f_-(0)}{2} = 0$$

Example: Let $f(t) = -1 - \frac{2t}{\pi}$ $\quad -\pi \leq t \leq 0$

$= -1 + \frac{2t}{\pi}$ $\quad 0 \leq t \leq \pi$

To solve $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = f(t)$

Find a fourier series for $f(t)$

$$f(t) = \sum_{-\infty,\, n\neq 0}^{\infty} \frac{2\{(-1)^n - 1\}}{\pi^2 n^2} e^{int}$$

Let $y(t) = \sum_{-\infty}^{\infty} y_n e^{int}$

$$\sum_{-\infty}^{\infty} [-n^2 + 2in + 2] y_n e^{int} = \sum_{-\infty,\, n\neq 0}^{\infty} \frac{2\{(-1)^n - 1\}}{\pi^2 n^2} e^{int}$$

$$\Rightarrow y_n = \frac{2((-1)^n - 1)}{\pi^2 n^2 (n^2 + 2 + 2in)}$$

$$\Rightarrow y = \sum_{-\infty}^{\infty} \frac{2((-1)^n - 1)}{\pi^2 n^2 (n^2 + 2in + 2)} e^{int}.$$

15.2 For a continuous function f,

$$G(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} f(t) e^{-i\omega t}\, dt$$

is called a fourier transform of f.

Then $F(t) = \int_{-\infty}^{\infty} G(\omega) e^{i\omega t}\, d\omega$

is called inverse fourier transform

- Sum of infinite sine's
- Position of $1/2\pi$

Example: $F(t) = \frac{1}{t^2+4}$; $G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{-i\omega t}}{t^2+4}\, dt$

$$G(\omega) = \frac{1}{2\pi}(-2\pi i)\, \mathrm{Res}\left(\frac{e^{-i\omega t}}{t^2+4}, -2i\right) = \frac{e^{-2\omega}}{4} \qquad \omega \geq 0$$

Contour in lower plane

and

$$G(\omega) = \frac{1}{2\pi}(2\pi i)\, \mathrm{Res}\left(\frac{e^{-i\omega t}}{t^2+4}, 2i\right) = \frac{e^{+2\omega}}{4}$$

$$G(\omega) = \frac{1}{4} e^{-2|\omega|}$$

Verify the inverse form.

$$\int_{-\infty}^{\infty} G(\omega) e^{i\omega t} d\omega = \int_{-\infty}^{\infty} \frac{e^{-2|\omega|}}{4} e^{i\omega t} d\omega = \frac{1}{t^2+4}$$

Example, $F(t) = 1, \quad -\pi \leq t \leq \pi$

$= 0 \quad$ otherwise

Then $G(\omega) = \frac{1}{2\pi} \int_{-\pi}^{\pi} (1) e^{-i\omega t} dt = \frac{\sin(\pi\omega)}{(\pi\omega)}$ (Diffraction)

Verify $F(t) = \int_{-\infty}^{\infty} \frac{\sin(\omega\pi)}{(\pi\omega)} e^{i\omega t} d\omega = 1 \quad |t| < \pi$

$= 0 \quad |t| > \pi$

$= \frac{1}{2} \quad t = \pm\pi$

5.3 Laplace transforms

Let $F(t)$ be defined on $[0, \infty)$ then

$$g(s) = \int_0^{\infty} f(t) e^{-st} dt$$

$$g = \mathcal{L}\{F\}$$

Example

(i) $\mathcal{L}\{e^{+at}\} = 1/(s-a) \qquad Re(s) > Re(a)$

(ii) $\mathcal{L}\{1\} = 1/s$

(iii) $\mathcal{L}\{\cos\omega t\} = Re\, \mathcal{L}\{e^{i\omega t}\} = \frac{s}{s^2+\omega^2} \qquad \omega$ (real)

$Re(s) > 0$

(iv) $\mathcal{L}(\sin\omega t) = Im\, \mathcal{L}\{e^{i\omega t}\} = \omega/(s^2+\omega^2) \qquad$ "

(v) $\mathcal{L}(\cosh\omega t) = \mathcal{L}(\cos(i\omega t)) = \frac{s}{s^2-\omega^2}$

(v) $\mathcal{L}(\sinh\omega t) = \mathcal{L}(-i\sin i\omega t) = \frac{\omega}{s^2-\omega^2}$

$\Big\}\ Re(s) > |\omega|$

(v) $\mathcal{L}\{F e^{-at}\}(s) = \mathcal{L}(f)(s+a)$

(vi) $\mathcal{L}\{aF(t) + bH(t)\} = a\mathcal{L}(F(t)) + b\mathcal{L}(H(t))$

(vi) $\mathcal{L}\{t^n\} = n!/s^{n+1}$

(vii) $\mathcal{L}\{t^n e^{at}\} = n!/(s-a)^{n+1}$

Remark: $\mathcal{L}(f)(s) = \int_0^\infty e^{-st} f(t)\, dt$

$$= f(t) \frac{e^{-st}}{(-s)} \Big|_0^\infty - \int_0^\infty f'(t) \frac{e^{-st}}{(-s)}\, dt$$

$$= \frac{1}{s} f(0) + \frac{1}{s} \mathcal{L}\{f'\}(s)$$

$$\Rightarrow \mathcal{L}\{f'\}(s) = s\mathcal{L}\{f\}(s) - f(0)$$

$$\Rightarrow \mathcal{L}\{f''\}(s) = s\mathcal{L}\{f'\}(s) - f'(0)$$

$$= s^2 \mathcal{L}\{f\}(s) - sf(0) - f'(0)$$

Example $f''(t) + 2f'(t) + f(t) = \sin t \qquad f(0) = 1, \ f'(0) = 0$

$$(s^2+2s+1)\mathcal{L}\{f\}(s) - (s+2) f(0) = \mathcal{L}\{\sin t\}(s) = 1/(s^2+1)$$

$$\Rightarrow \mathcal{L}\{f\}(s) = \frac{1}{(s+1)^2} \left[\frac{1}{s^2+1} + (s+2) \right]$$

$$\frac{s+2}{(s+1)^2} = \frac{1}{(s+1)} + \frac{1}{(s+1)^2}$$

$$\frac{1}{(s+1)^2(s^2+1)} = \frac{1}{2} \frac{1}{s+1} + \frac{1}{2} \frac{1}{(s+1)^2} - \frac{1}{2} \frac{s}{s^2+1}$$

$$\mathcal{L}\{f\}(s) = \frac{3}{2} \frac{1}{(s+1)} + \frac{3}{2} \frac{1}{(s+1)^2} - \frac{1}{2} \frac{s}{s^2+1}$$

$$\Rightarrow f(t) = \frac{3}{2} e^{-t} + \frac{3}{2} t e^{-t} - \frac{1}{2} \cos t$$

• Method of Green's Functions

Theorem: Consider $L = DpD + q$. Self adjoint operator with $\lambda = 0$, not an eigenvalue. Let

$$Lu = f(x) \qquad a \le x \le b$$

with
$$\left.\begin{array}{l} a_1 u(a) + a_2 u'(a) = 0 \\ b_1 u(b) + b_2 u'(b) = 0 \end{array}\right\} \text{Homogeneous b.c.}$$

p is possitive, p' is continuous on $[a,b]$

This system has a unique solⁿ

$$u(x) = \int_a^b G(x,t)\, \cancel{f}(t)\, dt$$

where $G(x,t)$ is called the Green's Function and is given by

$$G(x,t) = u_2(x) u_1(t) / p(\cancel{t}) W(u_1,u_2)(t) \qquad a \le t < x$$
$$= u_1(x) u_2(t) / p(t) W(u_1,u_2)(t) \qquad x < t \le b$$

where u_1 and u_2 are solⁿ of $Lu = 0$ with

$$a_1 u_1(a) + a_2 u_1'(a) = 0$$
$$\cancel{b_2} u_2(b) + b_2 u_2'(b) = 0$$

Proof: From theory of diff. eq. General solⁿ of $Lu = f$

$$u = c_1 u_1 + c_2 u_2 + u_p$$

u_p can be found by method of variation of parameters

$$u_p = v_1(x) u_1(x) + v_2(x) u_2(x)$$

$$v_1' = -\frac{f(x) u_2(x)}{PW} \qquad v_2' = \frac{f u_1}{PW}$$

since u_1 and u_2 are LI
$W \ne 0$
$p \ne 0$

$$u_p = u_1(\cancel{t}) \int_b^x (-) \frac{f(t) u_2(\cancel{x}t)}{PW} dt + u_2(x) \int_a^x \frac{f(\cancel{t}) u_1(t)}{PW} \cancel{dt}$$

$$= \int_a^b G(x,t)\, \cancel{f}(t)\, dt$$

Notion of Inverse operator: $(Tf)(x) = \int_a^b G(x,t) f(t) dt = u_p(x)$ ✓

$$L u_p(x) = f(x)$$

$$\Rightarrow \quad L^{-1} = T$$

if λ is ev of L then $Lu = f = \lambda u$

then $T^{-1}(f) = \lambda T^{-1}(u) = u$

$$\Rightarrow \; T^{-1}(u) = \tfrac{1}{\lambda} u.$$

$\Rightarrow$ $\tfrac{1}{\lambda}$ is ev of T.

- $L\,G(x,t) = \delta(x-t)$ clearly.

<u>Example</u> $y'' + y = f(x) = \operatorname{cosec} x$ say. $x \in [0, \pi/2]$

$y(0) = y(\pi/2) = 0$

$\Rightarrow$ $y'' + y = 0$ has two sol^{n} $y_1 = \sin(x)$ and $y_2 = \cos(x)$ which are LI and satisfy bc at 0 and π, respectively then

$W(y_1, y_2) = -\sin x \sin(x) - \cos^2 x = -1.$

$p(x) = 1$

Then $G(x,t) = -\cos(x)\sin t \qquad 0 \leq t < x$

$\qquad\qquad = -\sin(x)\cos t \qquad x < t \leq \pi/2$

So particular Sol^{n}

$$y_p = \int_0^{\pi/2} G(x,t)\, f(t)\, dt$$

$$\Rightarrow \quad \Rightarrow -\int_0^{x} \cos x \sin t \frac{1}{\sin t}\, dt = -x\cos x$$

$$\Rightarrow -\int_x^{\pi/2} \sin x \cot t\, dt = +\sin x \ln(\sin x)$$

$$y_p = -x\cos x + \sin x \ln(\sin x)$$