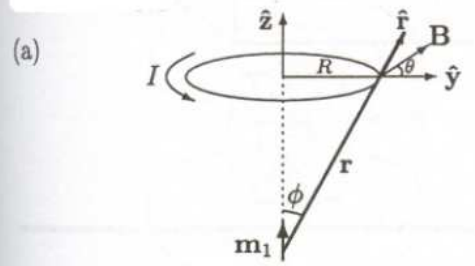
1. *[G 6.3] Find the force of attraction between two magnetic dipoles,* $\mathbf{m}_1$ *and* $\mathbf{m}_2$*, oriented as shown in the Fig., a distance* r *apart, (a) using* $F = 2\pi IRB\cos\theta$*, and (b) using* $\mathbf{F} = \nabla(\mathbf{m}\cdot\mathbf{B})$.



According to Eq. 6.2, $F = 2\pi IRB\cos\theta$. But $\mathbf{B} = \frac{\mu_0}{4\pi}\frac{[3(\mathbf{m}_1\cdot\hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{m}_1]}{r^3}$, and $B\cos\theta = \mathbf{B}\cdot\hat{\mathbf{y}}$, so $B\cos\theta = \frac{\mu_0}{4\pi}\frac{1}{r^3}[3(\mathbf{m}_1\cdot\hat{\mathbf{r}})(\hat{\mathbf{r}}\cdot\hat{\mathbf{y}}) - (\mathbf{m}_1\cdot\hat{\mathbf{y}})]$. But $\mathbf{m}_1\cdot\hat{\mathbf{y}} = 0$ and $\hat{\mathbf{r}}\cdot\hat{\mathbf{y}} = \sin\phi$, while $\mathbf{m}_1\cdot\hat{\mathbf{r}} = m_1\cos\theta$. $\therefore\ B\cos\theta = \frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1\sin\phi\cos\phi$.

$F = 2\pi IR\frac{\mu_0}{4\pi}\frac{1}{r^3}3m_1\sin\phi\cos\phi$. Now $\sin\phi = \frac{R}{r}$, $\cos\phi = \sqrt{r^2-R^2}/r$, so $F = 3\frac{\mu_0}{2}m_1IR^2\frac{\sqrt{r^2-R^2}}{r^5}$.

But $IR^2\pi = m_2$, so $F = \frac{3\mu_0}{2\pi}m_1m_2\frac{\sqrt{r^2-R^2}}{r^5}$, while for a dipole, $R \ll r$, so $\boxed{F = \frac{3\mu_0}{2\pi}\frac{m_1m_2}{r^4}}$.

(b) $\mathbf{F} = \nabla(\mathbf{m}_2\cdot\mathbf{B}) = (\mathbf{m}_2\cdot\nabla)\mathbf{B} = \left(m_2\frac{d}{dz}\right)\left[\frac{\mu_0}{4\pi}\frac{1}{z^3}\underbrace{(3(\mathbf{m}_1\cdot\hat{\mathbf{z}})\hat{\mathbf{z}} - \mathbf{m}_1)}_{2\mathbf{m}_1}\right] = \frac{\mu_0}{2\pi}m_1m_2\hat{\mathbf{z}}\underbrace{\frac{d}{dz}\left(\frac{1}{z^3}\right)}_{-3\frac{1}{z^4}}$,

or, since $z = r$: $\boxed{\mathbf{F} = -\frac{3\mu_0}{2\pi}\frac{m_1m_2}{r^4}\hat{\mathbf{z}}.}$

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2. *[G 6.8] A long circular cylinder of radius* R *carries a magnetization* $\mathbf{M} = ks^2\hat{\boldsymbol{\phi}}$*, where* k *is a constant,* s *is the distance from the axis, and* $\hat{\boldsymbol{\phi}}$ *is the usual azimuthal unit vector (Fig.). Find the magnetic field due to* $\mathbf{M}$*, for points inside and outside the cylinder.*

$$\nabla\times\mathbf{M} = \mathbf{J}_b = \frac{1}{s}\frac{\partial}{\partial s}(s\,ks^2)\hat{\mathbf{z}} = \frac{1}{s}(3ks^2)\hat{\mathbf{z}} = 3ks\hat{\mathbf{z}}, \quad \mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = ks^2(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{s}}) = -kR^2\hat{\mathbf{z}}.$$

So the bound current flows up the cylinder, and returns down the surface. [Incidentally, the *total* current should be zero ... *is* it? *Yes*, for $\int J_b\,da = \int_0^R (3ks)(2\pi s\,ds) = 2\pi kR^3$, while $\int K_b\,dl = (-kR^2)(2\pi R) = -2\pi kR^3$.] Since these currents have cylindrical symmetry, we can get the field by Ampère's law:

$$B\cdot 2\pi s = \mu_0 I_{\text{enc}} = \mu_0\int_0^s J_b\,da = 2\pi k\mu_0 s^3 \Rightarrow \boxed{\mathbf{B} = \mu_0 ks^2\hat{\boldsymbol{\phi}}} = \mu_0\mathbf{M}.$$

Outside the cylinder $I_{\text{enc}} = 0$, so $\boxed{\mathbf{B} = 0.}$

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3. *[G 6.12] An infinitely long cylinder, of radius* R*, carries a "frozen-in" magnetization, parallel to the axis,*

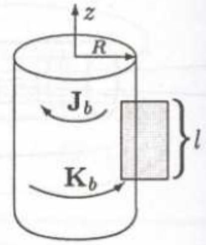
$$\mathbf{M} = ks\hat{\mathbf{z}},$$

where k *is a constant and* s *is the distance from the axis; there is no free current anywhere. Find the magnetic field inside and outside the cylinder by two different methods:*

(a) *Locate all bound currents, and calculate the field they produce.*

(b) *Use Ampere's law* $\oint \mathbf{H}\cdot d\mathbf{l} = I_{f_{enc}}$*, to find* $\mathbf{H}$*, and then get* $\mathbf{B}$ *from* $\mathbf{H} \equiv \frac{1}{\mu_0}\mathbf{B} - \mathbf{M}$.

(a) $\mathbf{M} = ks\hat{\mathbf{z}}$; $\mathbf{J}_b = \nabla\times\mathbf{M} = -k\hat{\boldsymbol{\phi}}$; $\mathbf{K}_b = \mathbf{M}\times\hat{\mathbf{n}} = kR\hat{\boldsymbol{\phi}}$.



B is in the z direction (this is essentially a superposition of solenoids). So $\boxed{\mathbf{B} = 0 \text{ outside.}}$ Use the amperian loop shown (shaded)—inner side at radius s:
$\oint \mathbf{B}\cdot d\mathbf{l} = Bl = \mu_0 I_{\text{enc}} = \mu_0\left[\int J_b\, da + K_b l\right] = \mu_0\left[-kl(R-s) + kRl\right] = \mu_0 kls$.
$\therefore \boxed{\mathbf{B} = \mu_0 ks\hat{\mathbf{z}} \text{ inside.}}$

(b) By symmetry, **H** points in the z direction. That same amperian loop gives $\oint \mathbf{H}\cdot d\mathbf{l} = Hl = \mu_0 I_{f_{\text{enc}}} = 0$, since there *is* no free current here. So $\boxed{\mathbf{H} = 0}$, and hence $\boxed{\mathbf{B} = \mu_0\mathbf{M}.}$ *Outside* $\mathbf{M} = 0$, so $\mathbf{B} = 0$; *inside* $\mathbf{M} = ks\hat{\mathbf{z}}$, so $\mathbf{B} = \mu_0 ks\hat{\mathbf{z}}$.

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4. *[G 7.11] A square loop is cut out of a thick sheet of aluminum. It is then placed so that the top portion is in a uniform magnetic field* **B**, *and allowed to fall under gravity (Fig.). (In the diagram, shading indicates the field region;* **B** *points into the page.) If the magnetic field is 1 T, find the terminal velocity of the loop (in m/s). Find the velocity of the loop as a function of time. How long does it take (in seconds) to reach, say, 90% of the terminal velocity? What would happen if you cut a tiny slit in the ring, breaking the circuit?*

$\mathcal{E} = Blv = IR \Rightarrow I = \frac{Bl}{R}v \Rightarrow$ upward magnetic force $= IlB = \frac{B^2l^2}{R}v$. This opposes the gravitational force downward:

$$mg - \frac{B^2l^2}{R}v = m\frac{dv}{dt};\quad \frac{dv}{dt} = g - \alpha v,\ \text{where } \alpha \equiv \frac{B^2l^2}{mR}.\quad g - \alpha v_t = 0 \Rightarrow v_t = \frac{g}{\alpha} = \boxed{\frac{mgR}{B^2l^2}}.$$

$$\frac{dv}{g-\alpha v} = dt \Rightarrow -\frac{1}{\alpha}\ln(g-\alpha v) = t + \text{const.} \Rightarrow g - \alpha v = Ae^{-\alpha t};\ \text{at } t=0,\ v=0,\ \text{so } A = g.$$

$$\alpha v = g(1-e^{-\alpha t});\quad v = \frac{g}{\alpha}(1-e^{-\alpha t}) = \boxed{v_t(1-e^{-\alpha t}).}$$

At 90% of terminal velocity, $v/v_t = 0.9 = 1 - e^{-\alpha t} \Rightarrow e^{-\alpha t} = 1 - 0.9 = 0.1$; $\ln(0.1) = -\alpha t$; $\ln 10 = \alpha t$; $t = \frac{1}{\alpha}\ln 10$, or $\boxed{t_{90\%} = \frac{v_t}{g}\ln 10.}$

Now the numbers: $m = 4\eta Al$, where η is the mass density of aluminum, A is the cross-sectional area, and l is the length of a side. $R = 4l/A\sigma$, where σ is the conductivity of aluminum. So

$$v_t = \frac{4\eta Alg4l}{A\sigma B^2l^2} = \frac{16\eta g}{\sigma B^2} = \frac{16g\eta\rho}{B^2},\ \text{and}\ \left\{\begin{array}{l}\rho = 2.8\times10^{-8}\,\Omega\,\text{m}\\ g = 9.8\,\text{m/s}^2\\ \eta = 2.7\times10^3\,\text{kg/m}^3\\ B = 1\,\text{T}\end{array}\right\}.$$

So $v_t = \frac{(16)(9.8)(2.7\times10^3)(2.8\times10^{-8})}{1} = \boxed{1.2\text{ cm/s};}$ $t_{90\%} = \frac{1.2\times10^{-2}}{9.8}\ln(10) = \boxed{2.8\text{ ms.}}$

$\boxed{\text{If the loop were cut, it would fall freely, with acceleration } g.}$

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5. *[G 7.12] A long solenoid of radius a, is driven by an alternating current, so that the field inside is sinusoidal:* $\mathbf{B}(t) = B_0\cos(\omega t)\,\hat{\mathbf{z}}$. *A circular loop of wire, of radius $a/2$ and resistance R, is placed inside the solenoid, and coaxial with it. Find the current induced in the loop, as a function of time.*

$$\Phi = \pi\left(\frac{a}{2}\right)^2 B = \frac{\pi a^2}{4}B_0\cos(\omega t);\ \mathcal{E} = -\frac{d\Phi}{dt} = \frac{\pi a^2}{4}B_0\omega\sin(\omega t).\quad I(t) = \frac{\mathcal{E}}{R} = \boxed{\frac{\pi a^2\omega}{4R}B_0\sin(\omega t).}$$

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6. *[G 7.18] A square loop, side a, resistance R, lies a distance s from an infinite straight wire that carries current I (Fig.). Now someone cuts the wire, so that I drops to zero. In what direction does the induced current in the square loop flow, and what total charge passes a given point in the loop during the time this current flows? If you don't like the scissors model, turn the current down gradually:*

$$I(t) = \begin{cases}(1-\alpha t)\,I, & for\,0 \le t \le 1/\alpha,\\ 0, & for\,t > 1/\alpha.\end{cases}$$

$$\Phi = \int \mathbf{B}\cdot d\mathbf{a};\ \mathbf{B} = \frac{\mu_0 I}{2\pi s}\hat{\phi};\ \Phi = \frac{\mu_0 I a}{2\pi}\int_a^{2a}\frac{ds}{s} = \frac{\mu_0 I a\ln 2}{2\pi};\ \mathcal{E} = I_{\text{loop}}R = \frac{dQ}{dt}R = -\frac{d\Phi}{dt} = -\frac{\mu_0 a\ln 2}{2\pi}\frac{dI}{dt}.$$

$$dQ = -\frac{\mu_0 a\ln 2}{2\pi R}dI \Rightarrow \boxed{Q = \frac{I\mu_0 a\ln 2}{2\pi R}}.$$

The field of the wire, at the square loop, is ***out of the page***, and ***decreasing***, so the field of the induced current must point out of page, within the loop, and hence the induced current flows $\boxed{\text{counterclockwise.}}$

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7. *[G 7.24] An alternating current $I_0\cos(\omega t)$ (amplitude 0.5 A, frequency 60 Hz) flows down a straight wire, which runs along the axis of a toroidal coil with rectangular cross section (inner radius 1 cm, outer radius 2 cm, height 1 cm, 1000 turns). The coil is connected to a 500 Ω resistor.*

   (a) *In the quasistatic approximation, what emf is induced in the toroid? Find the current, $I_r(t)$, in the resistor.*

   (b) *Calculate the back emf in the coil, due to the curent $I_r(t)$. What is the ratio of the amplitudes of this back emf and the "direct" emf in (a)?*

(a) In the quasistatic approximation $\mathbf{B} = \frac{\mu_0}{2\pi s}\hat{\phi}$. So $\Phi_1 = \frac{\mu_0 I}{2\pi}\int_a^b \frac{1}{s}h\,ds = \frac{\mu_0 I h}{2\pi}\ln(b/a)$.

This is the flux through *one* turn; the *total* flux is N times Φ_1: $\Phi = \frac{\mu_0 N h}{2\pi}\ln(b/a)I_0\cos(\omega t)$. So

$$\mathcal{E} = -\frac{d\Phi}{dt} = \frac{\mu_0 N h}{2\pi}\ln(b/a)I_0\omega\sin(\omega t) = \frac{(4\pi\times10^{-7})(10^3)(10^{-2})}{2\pi}\ln(2)(0.5)(2\pi\,60)\sin(\omega t)$$

$= \boxed{2.61\times10^{-4}\sin(\omega t)}$ (in volts), where $\omega = 2\pi\,60 = 377/\text{s}$. $I_r = \frac{\mathcal{E}}{R} = \frac{2.61\times10^{-4}}{500}\sin(\omega t)$

$= \boxed{5.22\times10^{-7}\sin(\omega t)}$ (amperes).

(b) $\mathcal{E}_b = -L\frac{dI_r}{dt}$; where (Eq. 7.27) $L = \frac{\mu_0 N^2 h}{2\pi}\ln(b/a) = \frac{(4\pi\times10^{-7})(10^6)(10^{-2})}{2\pi}\ln(2) = 1.39\times10^{-3}$ (henries).

Therefore $\mathcal{E}_b = -(1.39\times10^{-3})(5.22\times10^{-7}\,\omega)\cos(\omega t) = \boxed{-2.74\times10^{-7}\cos(\omega t)}$ (volts).

Ratio of amplitudes: $\frac{2.74\times10^{-7}}{2.61\times10^{-4}} = \boxed{1.05\times10^{-3}} = \frac{\mu_0 N^2 h\omega}{2\pi R}\ln(b/a)$.

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8. *[G 7.26] Find the energy stored in a section of length l of a long solenoid (radius R, current I, n turns per unit length), (a) using $W=\frac{1}{2}LI^2$; (b) using $W=\frac{1}{2}\oint(\mathbf{A}\cdot\mathbf{I})\,dl$; (c) using $W=\frac{1}{2\mu_0}\int_{allspace}B^2d\tau$ (d) using $W=\frac{1}{2\mu_0}\left[\int_{\mathcal{V}}B^2d\tau-\oint_S(\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}\right]$ (take as your volume the cylindrical tube from radius $a<R$ out to radius $b>R$).*

(a) $W=\frac{1}{2}LI^2$. $L=\mu_0 n^2\pi R^2 l$ (Prob. 7.22) $\boxed{W=\frac{1}{2}\mu_0 n^2\pi R^2 l I^2}$.

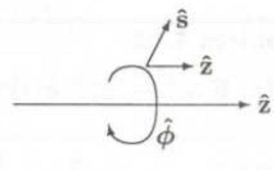
(b) $W=\frac{1}{2}\oint(\mathbf{A}\cdot\mathbf{I})dl$. $\mathbf{A}=(\mu_0 nI/2)R\,\hat{\boldsymbol{\phi}}$, at the surface (Eq. 5.70 or 5.71). So $W_1=\frac{1}{2}\frac{\mu_0 nI}{2}RI\cdot 2\pi R$, for one turn. There are nl such turns in length l, so $W=\frac{1}{2}\mu_0 n^2\pi R^2 l I^2$. ✓

(c) $W=\frac{1}{2\mu_0}\int B^2\,d\tau$. $B=\mu_0 nI$, inside, and zero outside; $\int d\tau=\pi R^2 l$, so $W=\frac{1}{2\mu_0}\mu_0^2 n^2 I^2\pi R^2 l=\frac{1}{2}\mu_0 n^2\pi R^2 l I^2$. ✓

(d) $W=\frac{1}{2\mu_0}\left[\int B^2\,d\tau-\oint(\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}\right]$. This time $\int B^2\,d\tau=\mu_0^2 n^2 I^2\pi(R^2-a^2)l$. Meanwhile, $\mathbf{A}\times\mathbf{B}=0$ outside (at $s=b$). Inside, $\mathbf{A}=\frac{\mu_0 nI}{2}a\,\hat{\boldsymbol{\phi}}$ (at $s=a$), while $\mathbf{B}=\mu_0 nI\,\hat{\mathbf{z}}$.

$$\mathbf{A}\times\mathbf{B}=\frac{1}{2}\mu_0^2 n^2 I^2 a\underbrace{(\hat{\boldsymbol{\phi}}\times\hat{\mathbf{z}})}_{\hat{\mathbf{s}}}$$

points *inward* ("out" of the volume)



$$\oint(\mathbf{A}\times\mathbf{B})\cdot d\mathbf{a}=\int\left(\frac{1}{2}\mu_0^2 n^2 I^2 a\hat{\mathbf{s}}\right)\cdot[a\,d\phi\,dz(-\hat{\mathbf{s}})]=-\frac{1}{2}\mu_0^2 n^2 I^2 a^2 2\pi l.$$

$$W=\frac{1}{2\mu_0}\left[\mu_0^2 n^2 I^2\pi(R^2-a^2)l+\mu_0^2 n^2 I^2\pi a^2 l\right]=\frac{1}{2}\mu_0 n^2 I^2 R^2\pi l.\ ✓$$

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9. *[G 7.32] Imagine* thin *wires connecting to the centers of plates as shown in Fig. (a), carrying constant current I. The radius of the capacitor is a and the separation of the plates is $w\ll a$. Assume that the current flows out over the plates in such a way that the surface charge is uniform, at any given time, and is zero at $t=0$.*

 (a) *Find the electric field between the plates, as a function of t.*

 (b) *Find the displacement current through a circle of radius s in the plane midway between the plates. Using this circle as your "Amperian loop", and the flat surface that spans it, find the magnetic field at a distance s from the axis.*

 (c) *Repeat part (b), but this time use the cylindrical surface in Fig. (b), which extends to the left through the plate and terminates outside the capacitor. Notice that the displacement current through this surface is zero, and there are two contributions to I_{enc}.*

(a) $\mathbf{E}=\frac{\sigma(t)}{\epsilon_0}\hat{\mathbf{z}}$; $\sigma(t)=\frac{Q(t)}{\pi a^2}=\frac{It}{\pi a^2}$; $\boxed{\frac{It}{\pi\epsilon_0 a^2}\hat{\mathbf{z}}.}$

(b) $I_{d_{enc}}=J_d\pi s^2=\epsilon_0\frac{dE}{dt}\pi s^2=\boxed{I\frac{s^2}{a^2}.}$ $\oint\mathbf{B}\cdot d\mathbf{l}=\mu_0 I_{d_{enc}}\Rightarrow B\,2\pi s=\mu_0 I\frac{s^2}{a^2}\Rightarrow\mathbf{B}=\boxed{\frac{\mu_0 I}{2\pi a^2}s\,\hat{\boldsymbol{\phi}}.}$

(c) A surface current flows radially outward over the left plate; let $I(s)$ be the total current crossing a circle of radius s. The charge density (at time t) is

$$\sigma(t)=\frac{[I-I(s)]t}{\pi s^2}.$$

Since we are told this is independent of s, it must be that $I-I(s)=\beta s^2$, for some constant β. But $I(a)=0$, so $\beta a^2=I$, or $\beta=I/a^2$. Therefore $I(s)=I(1-s^2/a^2)$.

$$B\,2\pi s=\mu_0 I_{enc}=\mu_0[I-I(s)]=\mu_0\frac{s^2}{a^2}\Rightarrow\boxed{\mathbf{B}=\frac{\mu_0}{2\pi a^2}s\,\hat{\boldsymbol{\phi}}.}$$

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10. *[G 7.34] Suppose*

$$\mathbf{E}(\mathbf{r},t)=-\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\theta(vt-r)\,\hat{\mathbf{r}};\mathbf{B}(\mathbf{r},t)=0$$

Show that these fields satisfy all of Maxwell's equations, and determine ρ and $\mathbf{J}$. Describe the physical situation that gives rise to these fields.

Physically, this is the field of a point charge $-q$ at the origin, out to an expanding spherical shell of radius vt; outside this shell the field is zero. Evidently the shell carries the opposite charge, $+q$. *Mathematically,* using product rule #5 and Eq. 1.99:

$$\nabla \cdot \mathbf{E} = \theta(vt-r)\nabla \cdot \left(-\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}}\right) - \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{\mathbf{r}} \cdot \nabla[\theta(vt-r)] = -\frac{q}{\epsilon_0}\delta^3(\mathbf{r})\theta(vt-r) - \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,(\hat{\mathbf{r}}\cdot\hat{\mathbf{r}})\,\frac{\partial}{\partial r}\theta(vt-r).$$

But $\delta^3(\mathbf{r})\theta(vt-r) = \delta^3(\mathbf{r})\theta(t)$, and $\frac{\partial}{\partial r}\theta(vt-r) = -\delta(vt-r)$ (Prob. 1.45), so

$$\rho = \epsilon_0 \nabla \cdot \mathbf{E} = \boxed{-q\delta^3(\mathbf{r})\theta(t) + \frac{q}{4\pi r^2}\delta(vt-r).}$$

(For $t < 0$ the field and the charge density are zero everywhere.)

Clearly $\nabla \cdot \mathbf{B} = 0$, and $\nabla \times \mathbf{E} = 0$ (since $\mathbf{E}$ has only an r component, and it is independent of θ and ϕ). There remains only the Ampére/Maxwell law, $\nabla \times \mathbf{B} = 0 = \mu_0\mathbf{J} + \mu_0\epsilon_0\partial\mathbf{E}/\partial t$. Evidently

$$\mathbf{J} = -\epsilon_0\frac{\partial \mathbf{E}}{\partial t} = -\epsilon_0\left\{-\frac{q}{4\pi\epsilon_0 r^2}\frac{\partial}{\partial t}\left[\theta(vt-r)\right]\right\}\hat{\mathbf{r}} = \boxed{\frac{q}{4\pi r^2}v\delta(vt-r)\,\hat{\mathbf{r}}.}$$

(The stationary charge at the origin does not contribute to $\mathbf{J}$, of course; for the expanding shell we have $\mathbf{J} = \rho\mathbf{v}$, as expected—Eq. 5.26.)
