1. A linear inhomogeneous dielectric is sandwiched between the plates of a parallel plate capacitor (separation between the plates $= d$) charged to the charge density σ. The permittivity of the dielectric at a distance y from one of the plates, is given by

$$\epsilon = \epsilon_0 \left(1 + K\left(\frac{y}{d}\right)\right)$$

where K is a positive constant. (Neglect edge effects.)

(a) Find the expressions for E, D, and P. Plot these quantities as a function of y.

(b) Find the bound charge densities σ_b and ρ_b. Plot ρ_b.

(c) Find the potential difference between the plates.

Solution

(a) First note that the $\mathbf{E}$, $\mathbf{D}$ and $\mathbf{P}$ are functions of y only and are in $\hat{\mathbf{y}}$ direction. Let the charge density on the plates be σ. Using Gauss law,

$$\begin{aligned} D(y) &= \sigma \\ E(y) &= D(y)/\epsilon(y) = \frac{\sigma}{\epsilon_0\left(1 + K\left(\frac{y}{d}\right)\right)} \\ P(y) &= D - \epsilon_0 E = \frac{\sigma K\left(\frac{y}{d}\right)}{\left(1 + K\left(\frac{y}{d}\right)\right)} \end{aligned}$$

(b) The bound surface charge density

$$\begin{aligned} \sigma_b(y=0) &= \mathbf{P}\cdot(-\hat{\mathbf{y}})|_{y=0} = 0 \\ \sigma_b(y=d) &= \mathbf{P}\cdot(\hat{\mathbf{y}})|_{y=d} = \frac{\sigma K}{1+K} \end{aligned}$$

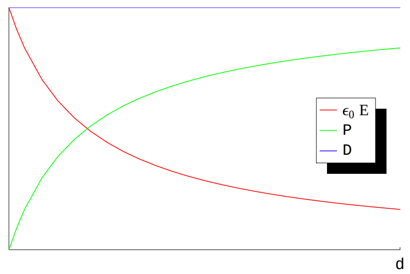
The bound volume charge density

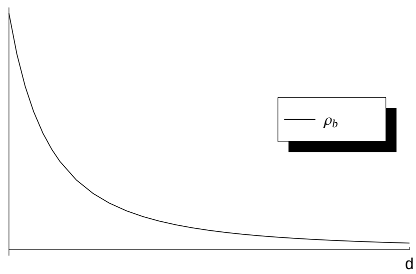
$$\begin{aligned} \rho_b(y) &= -\frac{d}{dy}P \\ &= -\frac{\sigma K}{d}\frac{1}{\left(1 + K\left(\frac{y}{d}\right)\right)^2} \end{aligned}$$

(c) The potential difference

$$\begin{aligned} V &= -\int_0^d E\,dy \\ &= -\frac{\sigma d}{\epsilon_0 K}\ln(1+K) \end{aligned}$$

The limiting value, as $K \to 0$, is $-\sigma d/\epsilon_0$, as expected.





2. A current is flowing in a thick wire of radius a. The current is distributed in the wire such that the current density at a distance r from the axis is given by

$$\mathbf{J} = \mathbf{J}_0 \left(1 + \frac{r^2}{a^2}\right).$$

Find the total current through the wire.

Solution:

Take a cross section of the wire that is perpendicular to the axis. Then, net current is

$$\begin{aligned} I &= \int_S \mathbf{J} \cdot d\mathbf{S} \\ &= \int_0^a \int_0^{2\pi} J_0 \left(1 + \frac{r^2}{a^2}\right) dr(r d\phi) \\ &= \frac{3}{2}\pi J_0 a^2 \end{aligned}$$

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3. Consider a wire, bent in a shape of a parabola, kept in XY plane with focus at origin. The disntace from apex to focus is d. The wire carries current I. Find the magnetic field at origin.

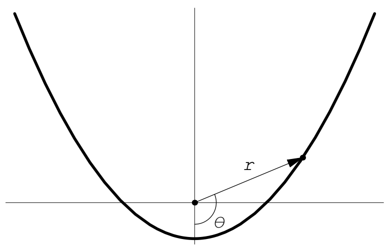
Solution:

Equation of the parabola: $r(1 + \cos\theta) = 2d$ with $\theta : -\pi \to \pi$
Let $\mathbf{r} = r\hat{\mathbf{r}}$ be the vector pointing to the parabola. The differntial tangent vector is given by

$$\begin{aligned} d\mathbf{l} = \left(\frac{d}{d\theta}\mathbf{r}\right) d\theta &= \left(\hat{\mathbf{r}}\frac{dr}{d\theta} + r\frac{d\hat{\mathbf{r}}}{d\theta}\right) d\theta \\ &= \left(\hat{\mathbf{r}}\frac{dr}{d\theta} + r\hat{\theta}\right) d\theta \end{aligned}$$

Then, $I\, d\mathbf{l} \times (0 - r\hat{\mathbf{r}}) = Ir^2 d\theta \hat{\mathbf{z}}$. Then the magnetic field at origin

$$\begin{aligned} \mathbf{B}(0) &= \frac{\mu_0}{4\pi} \int_{-\pi}^{\pi} \frac{Ir^2 \hat{\mathbf{z}}}{r^3} d\theta \\ &= \frac{\mu_0 I \hat{\mathbf{z}}}{4\pi} \int_{-\pi}^{\pi} \frac{1}{r} d\theta \\ &= \frac{\mu_0 I \hat{\mathbf{z}}}{8\pi d} \int_{-\pi}^{\pi} (1 + \cos\theta)\, d\theta \\ &= \frac{\mu_0 I}{4d} \hat{\mathbf{z}} \end{aligned}$$



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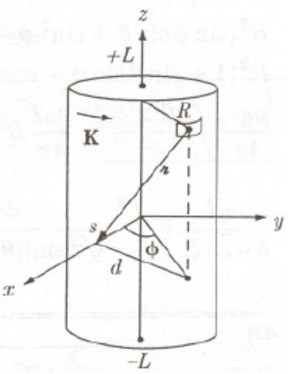
4. [G5.44] Use the Biot-Savart law to find the field inside and outside an infinitely long solenoid of radius R, with n turns per unit length, carrying a steady current I. [Write down the surface current density and Eq 5.39. Do z-integration first.]

Problem 5.44



Put the field point on the x axis, so $\mathbf{r} = (s, 0, 0)$. Then

$$\mathbf{B} = \frac{\mu_0}{4\pi}\int \frac{(\mathbf{K}\times\hat{\boldsymbol{\mathscr{r}}})}{\mathscr{r}^2}\,da;\quad da = R\,d\phi\,dz;\quad \mathbf{K} = K\,\hat{\boldsymbol{\phi}} =$$

$K(-\sin\phi\,\hat{\mathbf{x}} + \cos\phi\,\hat{\mathbf{y}}); \boldsymbol{\mathscr{r}} = (s - R\cos\phi)\,\hat{\mathbf{x}} - R\sin\phi\,\hat{\mathbf{y}} - z\,\hat{\mathbf{z}}.$

$$\mathbf{K}\times\boldsymbol{\mathscr{r}} = K\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ -\sin\phi & \cos\phi & 0\\ (s-R\cos\phi) & (-R\sin\phi) & (-z)\end{vmatrix} =$$

$K\left[(-z\cos\phi)\,\hat{\mathbf{x}} + (-z\sin\phi)\,\hat{\mathbf{y}} + (R - s\cos\phi)\,\hat{\mathbf{z}}\right];$

$\mathscr{r}^2 = z^2 + R^2 + s^2 - 2Rs\cos\phi$. The x and y components integrate to zero (z integrand is odd, as in Prob. 5.17).

$$\begin{aligned}
B_z &= \frac{\mu_0}{4\pi}KR\int\frac{(R - s\cos\phi)}{(z^2 + R^2 + s^2 - 2Rs\cos\phi)^{3/2}}\,d\phi\,dz\\
&= \frac{\mu_0 KR}{4\pi}\int_0^{2\pi}(R - s\cos\phi)\left\{\int_{-\infty}^{\infty}\frac{dz}{(z^2+d^2)^{3/2}}\right\}d\phi,
\end{aligned}$$

where $d^2 \equiv R^2 + s^2 - 2Rs\cos\phi$. Now $\displaystyle\int_{-\infty}^{\infty}\frac{dz}{(z^2+d^2)^{3/2}} = \left.\frac{2z}{d^2\sqrt{z^2+d^2}}\right|_0^{\infty} = \frac{2}{d^2}.$

$$\begin{aligned}
&= \frac{\mu_0 KR}{2\pi}\int_0^{2\pi}\frac{(R - s\cos\phi)}{(R^2+s^2-2Rs\cos\phi)}\,d\phi;\ (R - s\cos\phi) = \frac{1}{2R}\left[(R^2 - s^2) + (R^2 + s^2 - 2Rs\cos\phi)\right].\\
&= \frac{\mu_0 K}{4\pi}\left[(R^2 - s^2)\int_0^{2\pi}\frac{d\phi}{(R^2+s^2-2Rs\cos\phi)} + \int_0^{2\pi}d\phi\right].
\end{aligned}$$

$$\begin{aligned}
\int_0^{2\pi}\frac{d\phi}{a + b\cos\phi} &= 2\int_0^{\pi}\frac{d\phi}{a+b\cos\phi} = \frac{4}{\sqrt{a^2-b^2}}\tan^{-1}\left[\frac{\sqrt{a^2-b^2}\tan(\phi/2)}{a+b}\right]\Bigg|_0^{\pi}\\
&= \frac{4}{\sqrt{a^2-b^2}}\tan^{-1}\left[\frac{\sqrt{a^2-b^2}\tan(\pi/2)}{a+b}\right] = \frac{4}{\sqrt{a^2-b^2}}\left(\frac{\pi}{2}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}.
\end{aligned}$$

Here $a = R^2 + s^2$, $b = -2Rs$, so $a^2 - b^2 = R^4 + 2R^2s^2 + s^4 - 4R^2s^2 = R^4 - 2R^2s^2 + s^4 = (R^2 - s^2)^2$; $\sqrt{a^2 - b^2} = |R^2 - d^2|$.

$$B_z = \frac{\mu_0 K}{4\pi}\left[\frac{(R^2 - s^2)}{|R^2 - s^2|}2\pi + 2\pi\right] = \frac{\mu_0 K}{2}\left(\frac{R^2 - s^2}{|R^2 - s^2|} + 1\right).$$

Inside the solenoid, $s < R$, so $B_z = \frac{\mu_0 K}{2}(1+1) = \mu_0 K$. *Outside* the solenoid, $s > R$, so $B_z = \frac{\mu_0 K}{2}(-1+1) = 0$. Here $K = nI$, so $\boxed{\mathbf{B} = \mu_0 nI\,\hat{\mathbf{z}}\text{(inside), and 0(outside)}}$ (as we found more easily using Ampére's law, in Ex. 5.9).

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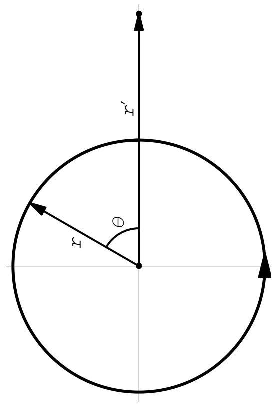
5. Consider a circular ring, of radius R and carrying current I is placed in the XY plane with its center at origin. Set up the integral to find the magnetic field at a point on the X axis, at a distance $d\,(\gg R)$ from the origin. Now, expand the integrand in the powers of R/d and find the first non-zero term of the field. Express in terms of $m = I(\pi R^2)$.

Solution:

- $\mathbf{r}' = d\hat{\mathbf{x}}$
- $\mathbf{r} = R\hat{\mathbf{s}} = R(\cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}})$
- $d\mathbf{l} = Rd\phi\,\hat{\phi} = Rd\phi(-\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}})$
- $\left|\mathbf{r}' - \mathbf{r}\right| = (d^2 + R^2 - 2Rd\cos\phi)^{1/2}$
- $Id\mathbf{l} \times \left(\mathbf{r}' - \mathbf{r}\right) = IR(R - d\cos\phi)\,d\phi\hat{\mathbf{z}}$



Then the magnetic field at $\mathbf{r}'$ is

$$\mathbf{B}(\mathbf{r}') = \frac{\mu_0}{4\pi}\int_0^{2\pi} \frac{IR(R - d\cos\phi)\,d\phi\hat{\mathbf{z}}}{(d^2 + R^2 - 2Rd\cos\phi)^{3/2}}$$

Now use,

$$\begin{aligned}
\frac{1}{|\mathbf{r}' - \mathbf{r}|^3} &= \left[\frac{1}{(d^2 + R^2 - 2Rd\cos\phi)^{1/2}}\right]^3 \\
&= \frac{1}{d^3}\left[\sum_{n=0}^{\infty}\left(\frac{R}{d}\right)^n P_n(\cos\phi)\right]^3 \\
&\approx \frac{1}{d^3}\left[1 + 3\frac{R}{d}\cos\phi + \mathcal{O}\left(\left(\frac{R}{d}\right)^2\right)\right]
\end{aligned}$$

Then,

$$\begin{aligned}
\mathbf{B}(\mathbf{r}') &= \frac{\mu_0 IR\hat{\mathbf{z}}}{4\pi d^2}\int_0^{2\pi}\left(\frac{R}{d} - \cos\phi\right)\left[1 + 3\frac{R}{d}\cos\phi\right]d\phi \\
&\approx \frac{\mu_0 IR\hat{\mathbf{z}}}{4\pi d^2}\int_0^{2\pi}\left(-\cos\phi + \frac{R}{d}\left(1 - 3\cos^2\phi\right)\right)d\phi \\
&= -\frac{\mu_0\left(I\pi R^2\right)}{4\pi d^3}\hat{\mathbf{z}}
\end{aligned}$$

This is consistent with the dipole field in the direction perpendicular to the direction of the dipole.

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6. [G5.6]

(a) A phonograph record carries a uniform density of "static electricity" σ. If it rotates at angular velocity ω, what is the surface current density $\mathbf{K}$ at a distance r from the center?

(b) A uniformly charged solid sphere, of radius R and total charge Q, is centered about origin and spinning at a constant angular velocity ω about the z axis. Find the current density $\mathbf{J}$ at any point (r, θ, ϕ) within the sphere.

Solution:
(a) $v = \omega r$, so $\boxed{K = \sigma\omega r.}$ (b) $\mathbf{v} = \omega r\sin\theta\,\hat{\boldsymbol{\phi}} \Rightarrow \boxed{\mathbf{J} = \rho\omega r\sin\theta\,\hat{\boldsymbol{\phi}},}$ where $\rho \equiv Q/(4/3)\pi R^3$.

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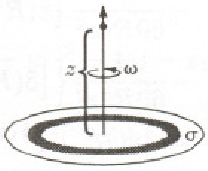
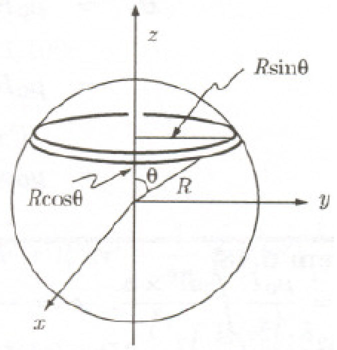
7. [G5.47] Find the magnetic field at a point $z > R$ on the axis of (a) the rotating disk and (b) the rotating sphere, of problem G5.6



(a) The total charge on the shaded ring is $dq = \sigma(2\pi r)\,dr$. The time for one revolution is $dt = 2\pi/\omega$. So the current in the ring is $I = \frac{dq}{dt} = \sigma\omega r\,dr$. From Eq. 5.38, the magnetic field of this ring (for points on the axis) is $d\mathbf{B} = \frac{\mu_0}{2}\sigma\omega r\frac{r^2}{(r^2+z^2)^{3/2}}\,dr\,\hat{\mathbf{z}}$, and the total field of the disk is

$$\mathbf{B} = \frac{\mu_0\sigma\omega}{2}\int_0^R \frac{r^3\,dr}{(r^2+z^2)^{3/2}}\,\hat{\mathbf{z}}. \quad \text{Let } u \equiv r^2, \text{ so } du = 2r\,dr. \quad \text{Then}$$

$$= \frac{\mu_0\sigma\omega}{4}\int_0^{R^2}\frac{u\,du}{(u+z^2)^{3/2}} = \frac{\mu_0\sigma\omega}{4}\left[2\left(\frac{u+2z^2}{\sqrt{u+z^2}}\right)\right]\Bigg|_0^{R^2} = \boxed{\frac{\mu_0\sigma\omega}{2}\left[\frac{(R^2+2z^2)}{\sqrt{R^2+z^2}} - 2z\right]\hat{\mathbf{z}}.}$$



(b) Slice the sphere into slabs of thickness t, and use (a). Here $t = |d(R\cos\theta)| = R\sin\theta\,d\theta$; $\sigma \to \rho t = \rho R\sin\theta\,d\theta$; $R \to R\sin\theta$; $z \to z - R\cos\theta$. First rewrite the term in square brackets:

$$\left[\frac{(R^2+2z^2)}{\sqrt{R^2+z^2}} - 2z\right] = \frac{2(R^2+z^2)}{\sqrt{R^2+z^2}} - \frac{R^2}{\sqrt{R^2+z^2}} - 2z$$
$$= 2\left[\sqrt{R^2+z^2} - \frac{R^2/2}{\sqrt{R^2+z^2}} - z\right].$$

But $R^2+z^2 \to R^2\sin^2\theta + (z^2 - 2Rz\cos\theta + R^2\cos^2\theta) = R^2 + z^2 - 2Rz\cos\theta$. So

$$B_z = \frac{\mu_0\rho R\omega}{2}2\int_0^\pi \sin\theta\,d\theta\left[\sqrt{R^2+z^2-2Rz\cos\theta} - \frac{(R^2/2)\sin^2\theta}{\sqrt{R^2+z^2-2Rz\cos\theta}} - (z - R\cos\theta)\right].$$

Let $u \equiv \cos\theta$, so $du = -\sin\theta\,d\theta$; $\theta: 0 \to \pi \Rightarrow u: 1 \to -1$; $\sin^2\theta = 1 - u^2$.

$$= \mu_0\rho R\omega\int_{-1}^1\left[\sqrt{R^2+z^2-2Rzu} - \frac{(R^2/2)(1-u^2)}{\sqrt{R^2+z^2-2Rzu}} - z + Ru\right]du$$
$$= \mu_0\rho R\omega\left[I_1 - \frac{R^2}{2}(I_2 - I_3) - I_4 + I_5\right].$$

$$I_1 = \int_{-1}^1\sqrt{R^2+z^2-2Rzu}\,du = -\frac{1}{3Rz}\left(R^2+z^2-2Rzu\right)^{3/2}\Big|_{-1}^1$$
$$= -\frac{1}{3Rz}\left[\left(R^2+z^2-2Rz\right)^{3/2} - \left(R^2+z^2+2Rz\right)^{3/2}\right] = -\frac{1}{3Rz}\left[(z-R)^3 - (z+R)^3\right]$$
$$= -\frac{1}{3Rz}\left(z^3 - 3z^2R + 3zR^2 - R^3 - z^3 - 3z^2R - 3zR^2 - R^3\right) = \frac{2}{3z}(3z^2 + R^2).$$

$$I_2 = \int_{-1}^1\frac{1}{\sqrt{R^2+z^2-2Rzu}}\,du = -\frac{1}{Rz}\sqrt{R^2+z^2-2Rzu}\Big|_{-1}^1 = -\frac{1}{Rz}\left[(z-R) - (z+R)\right] = \frac{2}{z}.$$

$$\begin{aligned}
I_3 &= \int_{-1}^{1} \frac{u^2}{\sqrt{R^2+z^2-2Rzu}}\,du \\
&= -\frac{1}{60R^3z^3}\left[8(R^2+z^2)^2+4(R^2+z^2)2Rzu+3(2Rz)^2u^2\right]\sqrt{R^2+z^2-2Rzu}\Big|_{-1}^{1} \\
&= -\frac{1}{60R^3z^3}\Big\{\left[8(R^2+z^2)^2+8Rz(R^2+z^2)+12R^2z^2\right](z-R) \\
&\quad -\left[8(R^2+z^2)^2-8Rz(R^2+z^2)+12R^2z^2\right](z+R)\Big\} \\
&= -\frac{1}{60R^3z^3}\left\{z\left[16Rz(R^2+z^2)\right]-R\left[16(R^2+z^2)^2+24R^2z^2\right]\right\} \\
&= -\frac{1}{60R^3z^3}16R\left(R^2z^2+z^4-R^4-2R^2z^2-z^4-\frac{3}{2}R^2z^2\right) \\
&= -\frac{4}{15R^2z^3}\left(-\frac{5}{2}R^2z^2-R^4\right)=\frac{4}{15z^3}\left(R^2+\frac{5}{2}z^2\right). \quad I_4 = z\int_{-1}^{1}du = 2z; \quad I_5 = R\int_{-1}^{1}u\,du = 0.
\end{aligned}$$

$$\begin{aligned}
B_z &= \mu_0 R\rho\omega\left[\frac{2}{3z}(3z^2+R^2)-\frac{R^2}{2}\frac{2}{z}+\frac{R^2}{2}\frac{4}{15z^3}\left(R^2+\frac{5}{2}z^2\right)-2z\right] \\
&= \mu_0 R\rho\omega\left(2z+\frac{2R^2}{3z}-\frac{R^2}{z}+\frac{2R^4}{15z^3}+\frac{R^2}{3z}-2z\right) \\
&= \mu_0\rho\omega\frac{2R^5}{15z^3}. \quad \text{But } \rho = \frac{Q}{(4/3)\pi R^3}, \text{ so } \boxed{\mathbf{B} = \frac{\mu_0 Q\omega R^2}{10\pi z^3}\,\hat{\mathbf{z}}.}
\end{aligned}$$