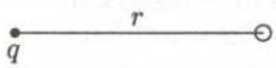
1. *[G 4.4] A point charge q is situated at a large distance r from a neutral atom of polarizability α. Find the force of attraction between them.*



Field of q: $\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}$. Induced dipole moment of atom: $\mathbf{p} = \alpha\mathbf{E} = \frac{\alpha q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$.

Field of this dipole, at location of q ($\theta = \pi$, in Eq. 3.103): $E = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left(\frac{2\alpha q}{4\pi\epsilon_0 r^2}\right)$ (to the right).

Force on q due to this field: $\boxed{F = 2\alpha\left(\frac{q}{4\pi\epsilon_0}\right)^2\frac{1}{r^3}}$ (attractive).

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2. *Consider a localized (of small dimension) charge distribution ρ with zero net charge and dipole moment $\mathbf{p}$, placed in an external field $\mathbf{E}_{ext}$. Let 0 be some suitable origin.*

   (a) *Show that the force on the charge distribution is given by*

$$\mathbf{F} = (\mathbf{p}\cdot\nabla)\,\mathbf{E}_{ext}(0) + \cdots$$

   (b) *Show that the torque on the charge distribution is given by*

$$\mathbf{N} = \mathbf{p}\times\mathbf{E}_{ext}(0) + \cdots$$

   (c) *Show that the energy of the charge distribution is given by*

$$U = -\mathbf{p}\cdot\mathbf{E}_{ext}$$

Let 0 be the origin. Assume that the charge distribution is small enough such that the variation of $\mathbf{E}$ over the dimension of the charge distribution is slow.

(a) The x component of the net force on the charge distribution (due to $\mathbf{E}_{\text{ext}}$) is

$$F_x = \int_V \rho(\mathbf{r})\, E_{\text{ext},x}(\mathbf{r})\, dv$$

Using Taylor expansion

$$\begin{aligned} F_x &= \int_V \rho(\mathbf{r})\left[E_{\text{ext},x}(0) + \mathbf{r}\cdot\nabla E_{\text{ext},x}(0) + \cdots\right] dv \\ &= \left[E_{\text{ext},x}(0)\int_V \rho(\mathbf{r})\, dv + \nabla E_{\text{ext},x}(0)\cdot\int_V \mathbf{r}\rho(\mathbf{r})\, dv + \cdots\right] \\ &= 0 + \mathbf{p}\cdot\nabla E_{\text{ext},x}(0) \end{aligned}$$

(b) The net torque is

$$\mathbf{T} = \int_V \rho(\mathbf{r})\ \mathbf{r}\times\mathbf{E}_{\text{ext}}(\mathbf{r})\, dv$$

Keeping only first term is the Taylor expansion of $\mathbf{E}$, we get

$$\begin{aligned} \mathbf{T} &= \int_V \rho(\mathbf{r})\ \mathbf{r}\times\mathbf{E}_{\text{ext}}(0)\, dv + \cdots \\ &= \left(\int_V \rho(\mathbf{r})\ \mathbf{r} dv\right)\times\mathbf{E}_{\text{ext}}(0) + \cdots \\ &= \mathbf{p}\times\mathbf{E}_{\text{ext}}(0) + \cdots \end{aligned}$$

(c) Simillarly, the potential energy

$$\begin{aligned} U &= \int_V \rho(\mathbf{r}) V(\mathbf{r}) dv \\ &= \int_V \rho(\mathbf{r}) \left(V(0) + \mathbf{r} \cdot \nabla V(0) + \cdots \right) dv \\ &= 0 - \mathbf{p} \cdot \mathbf{E} + \cdots \end{aligned}$$

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3. *[G 4.5, G 4.29] In Fig.,* $\mathbf{p_1}$ *and* $\mathbf{p_2}$ *are (perfect) dipoles a distance* r *apart. What is the torque on* p_1 *due to* p_2*? What is the torque on* p_2 *due to* p_1*?*
For the same configuration, calculate the force on $\mathbf{p_2}$ *due to* $\mathbf{p_1}$*, and the force on* $\mathbf{p_1}$ *due to* $\mathbf{p_2}$*. Are the answers consistent with Newton's third law?*
Also, find the total torque on $\mathbf{p_2}$ *with respect to the center of* $\mathbf{p_1}$*, and compare it with the torque on* $\mathbf{p_1}$ *about that same point.*

Field of $\mathbf{p}_1$ at $\mathbf{p}_2$ ($\theta = \pi/2$ in Eq. 3.103): $\mathbf{E}_1 = \dfrac{p_1}{4\pi\epsilon_0 r^3}\,\hat{\boldsymbol{\theta}}$ (points *down*).

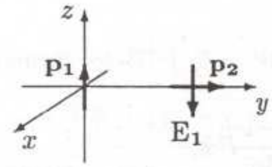
Torque on $\mathbf{p}_2$: $\mathbf{N}_2 = \mathbf{p}_2 \times \mathbf{E}_1 = p_2 E_1 \sin 90° = p_2 E_1 = \boxed{\dfrac{p_1 p_2}{4\pi\epsilon_0 r^3}}$ (points *into* the page).

Field of $\mathbf{p}_2$ at $\mathbf{p}_1$ ($\theta = \pi$ in Eq. 3.103): $\mathbf{E}_2 = \dfrac{p_2}{4\pi\epsilon_0 r^3}(-2\,\hat{\mathbf{r}})$ (points to the *right*).

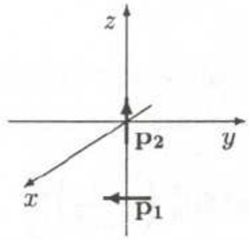
Torque on $\mathbf{p}_1$: $\mathbf{N}_1 = \mathbf{p}_1 \times \mathbf{E}_2 = \boxed{\dfrac{2p_1 p_2}{4\pi\epsilon_0 r^3}}$ (points *into* the page).

(a) Eq. 4.5 $\Rightarrow \mathbf{F}_2 = (\mathbf{p}_2 \cdot \boldsymbol{\nabla})\,\mathbf{E}_1 = p_2 \frac{\partial}{\partial y}(\mathbf{E}_1)$;

Eq. 3.103 $\Rightarrow \mathbf{E}_1 = \frac{p_1}{4\pi\epsilon_0 r^3}\,\hat{\boldsymbol{\theta}} = -\frac{p_1}{4\pi\epsilon_0 y^3}\,\hat{\mathbf{z}}$. Therefore



$$\mathbf{F}_2 = -\frac{p_1 p_2}{4\pi\epsilon_0}\left[\frac{d}{dy}\left(\frac{1}{y^3}\right)\right]\hat{\mathbf{z}} = \frac{3p_1p_2}{4\pi\epsilon_0 y^4}\,\hat{\mathbf{z}}, \text{ or } \boxed{\mathbf{F}_2 = \frac{3p_1p_2}{4\pi\epsilon_0 r^4}\,\hat{\mathbf{z}}} \text{ (upward).}$$



To calculate $\mathbf{F}_1$, put $\mathbf{p}_2$ at the origin, pointing in the z direction; then $\mathbf{p}_1$ is at $-r\,\hat{\mathbf{z}}$, and it points in the $-\hat{\mathbf{y}}$ direction. So $\mathbf{F}_1 = (\mathbf{p}_1 \cdot \boldsymbol{\nabla})\,\mathbf{E}_2 = -p_1 \left.\frac{\partial \mathbf{E}_2}{\partial y}\right|_{x=y=0,\,z=-r}$; we need $\mathbf{E}_2$ as a function of x, y, and z.

From Eq. 3.104: $\mathbf{E}_2 = \frac{1}{4\pi\epsilon_0}\frac{1}{r^3}\left[\frac{3(\mathbf{p}_2\cdot\mathbf{r})\mathbf{r}}{r^2} - \mathbf{p}\right]$, where $\mathbf{r} = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}$, $\mathbf{p}_2 = -p_2\,\hat{\mathbf{y}}$, and hence $\mathbf{p}_2 \cdot \mathbf{r} = -p_2 y$.

$$\mathbf{E}_2 = \frac{p_2}{4\pi\epsilon_0}\left[\frac{-3y(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) + (x^2+y^2+z^2)\,\hat{\mathbf{y}}}{(x^2+y^2+z^2)^{5/2}}\right] = \frac{p_2}{4\pi\epsilon_0}\left[\frac{-3xy\,\hat{\mathbf{x}} + (x^2-2y^2+z^2)\,\hat{\mathbf{y}} - 3yz\,\hat{\mathbf{z}}}{(x^2+y^2+z^2)^{5/2}}\right]$$

$$\frac{\partial \mathbf{E}_2}{\partial y} = \frac{p_2}{4\pi\epsilon_0}\left\{-\frac{5}{2}\frac{1}{r^7}2y[-3xy\,\hat{\mathbf{x}} + (x^2-2y^2+z^2)\,\hat{\mathbf{y}} - 3yz\,\hat{\mathbf{z}}] + \frac{1}{r^5}(-3x\,\hat{\mathbf{x}} - 4y\,\hat{\mathbf{y}} - 3z\,\hat{\mathbf{z}})\right\};$$

$$\left.\frac{\partial \mathbf{E}_2}{\partial y}\right|_{(0,0)} = \frac{p_2}{4\pi\epsilon_0}\frac{-3z}{r^5}\,\hat{\mathbf{z}};\quad \mathbf{F}_1 = -p_1\left(\frac{p_2}{4\pi\epsilon_0}\frac{3r}{r^5}\,\hat{\mathbf{z}}\right) = \boxed{-\frac{3p_1p_2}{4\pi\epsilon_0 r^4}\,\hat{\mathbf{z}}.}$$

These results *are* consistent with Newton's third law: $\mathbf{F}_1 = -\mathbf{F}_2$.

(b) $\mathbf{N}_2 = (\mathbf{p}_2 \times \mathbf{E}_1) + (\mathbf{r} \times \mathbf{F}_2)$. The first term was calculated in Prob. 4.5; the second we get from (a), using $\mathbf{r} = r\,\hat{\mathbf{y}}$:

$$\mathbf{p}_2 \times \mathbf{E}_1 = \frac{p_1p_2}{4\pi\epsilon_0 r^3}(-\hat{\mathbf{x}});\quad \mathbf{r} \times \mathbf{F}_2 = (r\,\hat{\mathbf{y}}) \times \left(\frac{3p_1p_2}{4\pi\epsilon_0 r^4}\,\hat{\mathbf{z}}\right) = \frac{3p_1p_2}{4\pi\epsilon_0 r^3}\,\hat{\mathbf{x}};\text{ so } \boxed{\mathbf{N}_2 = \frac{2p_1p_2}{4\pi\epsilon_0 r^3}\,\hat{\mathbf{x}}.}$$

This is equal and opposite to the torque on $\mathbf{p}_1$ due to $\mathbf{p}_2$, with respect to the center of $\mathbf{p}_1$ (see Prob. 4.5).

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4. *[G 4.13] A very long cylinder, of radius a, carries a uniform polarization* $\mathbf{P}$ *perpendicular to its axis. Find the electric field inside the cylinder. Show that the field outside the cylinder can be expressed in the form*

$$\mathbf{E}(\mathbf{r}) = \frac{a^2}{2\epsilon_0 s^2}\left[2(\mathbf{P}\cdot\hat{\mathbf{s}})\,\hat{\mathbf{s}} - \mathbf{P}\right].$$

Think of it as two cylinders of opposite uniform charge density $\pm\rho$. *Inside*, the field at a distance s from the axis of a uniformly charge cylinder is given by Gauss's law: $E2\pi s\ell = \frac{1}{\epsilon_0}\rho\pi s^2\ell \Rightarrow \mathbf{E} = (\rho/2\epsilon_0)\mathbf{s}$. For *two* such cylinders, one plus and one minus, the net field (inside) is $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = (\rho/2\epsilon_0)(\mathbf{s}_+ - \mathbf{s}_-)$. But $\mathbf{s}_+ - \mathbf{s}_- = -\mathbf{d}$, so $\mathbf{E} = \boxed{-\rho\mathbf{d}/(2\epsilon_0),}$ where $\mathbf{d}$ is the vector from the negative axis to positive axis. In this case the total dipole moment of a chunk of length ℓ is $\mathbf{P}(\pi a^2\ell) = (\rho\pi a^2\ell)\,\mathbf{d}$. So $\rho\mathbf{d} = \mathbf{P}$, and $\boxed{\mathbf{E} = -\mathbf{P}/(2\epsilon_0),}$ for $s < a$.

Outside, Gauss's law gives $E2\pi s\ell = \frac{1}{\epsilon_0}\rho\pi a^2\ell \Rightarrow \mathbf{E} = \frac{\rho a^2}{2\epsilon_0}\frac{\hat{\mathbf{s}}}{s}$, for *one* cylinder. For the combination, $\mathbf{E} = \mathbf{E}_+ + \mathbf{E}_- = \frac{\rho a^2}{2\epsilon_0}\left(\frac{\hat{\mathbf{s}}_+}{s_+} - \frac{\hat{\mathbf{s}}_-}{s_-}\right)$, where

$$\begin{aligned}
\mathbf{s}_\pm &= \mathbf{s} \mp \frac{\mathbf{d}}{2}; \\
\frac{\mathbf{s}_\pm}{s_\pm^2} &= \left(\mathbf{s} \mp \frac{\mathbf{d}}{2}\right)\left(s^2 + \frac{d^2}{4} \mp \mathbf{s}\cdot\mathbf{d}\right)^{-1} \cong \frac{1}{s^2}\left(\mathbf{s} \mp \frac{\mathbf{d}}{2}\right)\left(1 \mp \frac{\mathbf{s}\cdot\mathbf{d}}{s^2}\right)^{-1} \cong \frac{1}{s^2}\left(\mathbf{s} \mp \frac{\mathbf{d}}{2}\right)\left(1 \pm \frac{\mathbf{s}\cdot\mathbf{d}}{s^2}\right) \\
&= \frac{1}{s^2}\left(\mathbf{s} \pm \mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2} \mp \frac{\mathbf{d}}{2}\right) \text{ (keeping only 1st order terms in } \mathbf{d}\text{).} \\
\left(\frac{\hat{\mathbf{s}}_+}{s_+} - \frac{\hat{\mathbf{s}}_-}{s_-}\right) &= \frac{1}{s^2}\left[\left(\mathbf{s} + \mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2} - \frac{\mathbf{d}}{2}\right) - \left(\mathbf{s} - \mathbf{s}\frac{(\mathbf{s}\cdot\mathbf{d})}{s^2} + \frac{\mathbf{s}}{2}\right)\right] = \frac{1}{s^2}\left(2\frac{\mathbf{s}(\mathbf{s}\cdot\mathbf{d})}{s^2} - \mathbf{d}\right).
\end{aligned}$$

$$\boxed{\mathbf{E}(\mathbf{s}) = \frac{a^2}{2\epsilon_0}\frac{1}{s^2}\left[2(\mathbf{P}\cdot\hat{\mathbf{s}})\,\hat{\mathbf{s}} - \mathbf{P}\right],} \quad \text{for } s > a.$$

---

5. *[G 4.15] A thick spherical shell (inner radius a, outer radius b) is made of dielectric material with a "frozen-in" polarization*

$$\mathbf{P}(\mathbf{r}) = \frac{k}{r}\hat{\mathbf{r}},$$

where k is a constant and r is the distance from the center (Fig.). (There is no free *charge in the problem.) Find the electric field in all three regions by two different methods:*

   (a) *Locate all the bound charge, and use Gauss's law to calculate the field it produces.*

   (b) *Use $\oint \mathbf{D}\cdot d\mathbf{a} = \mathbf{Q}_{f_{enc}}$, to find $\mathbf{D}$, and then get $\mathbf{E}$ from $\mathbf{D} \equiv \epsilon_0\mathbf{E} + \mathbf{P}$.*

(a) $\rho_b = -\nabla\cdot\mathbf{P} = -\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{k}{r}\right) = -\frac{k}{r^2}$; $\quad \sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \begin{cases} +\mathbf{P}\cdot\hat{\mathbf{r}} = k/b & (\text{at } r = b), \\ -\mathbf{P}\cdot\hat{\mathbf{r}} = -k/a & (\text{at } r = a). \end{cases}$

Gauss's law $\Rightarrow \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{r^2}\hat{\mathbf{r}}$. For $r < a$, $Q_{\text{enc}} = 0$, so $\boxed{\mathbf{E} = 0.}$ For $r > b$, $Q_{\text{enc}} = 0$ (Prob. 4.14), so $\boxed{\mathbf{E} = 0.}$

For $a < r < b$, $Q_{\text{enc}} = \left(\frac{-k}{a}\right)(4\pi a^2) + \int_a^r \left(\frac{-k}{\bar{r}^2}\right)4\pi\bar{r}^2 d\bar{r} = -4\pi ka - 4\pi k(r-a) = -4\pi kr$; so $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}}.}$

(b) $\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{\text{enc}}} = 0 \Rightarrow \mathbf{D} = 0$ everywhere. $\mathbf{D} = \epsilon_0\mathbf{E} + \mathbf{P} = 0 \Rightarrow \mathbf{E} = (-1/\epsilon_0)\mathbf{P}$, so

$\boxed{\mathbf{E} = 0 \text{ (for } r < a \text{ and } r > b);}$ $\boxed{\mathbf{E} = -(k/\epsilon_0 r)\,\hat{\mathbf{r}} \text{ (for } a < r < b).}$

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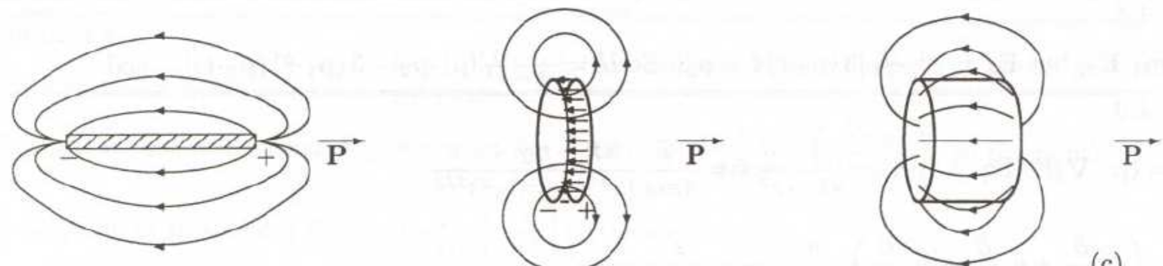
6. *[G 4.11] A short cylinder, of radius a and length L, carries a "frozen-in" uniform polarization $\mathbf{P}$, parallel to its axis. Find the bound charge, and sketch the electric field (i) for $L >> a$, (ii) $L << a$ and (iii) $L \approx a$.*

$\rho_b = 0$; $\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}} = \pm P$ (plus sign at one end—the one **P** points *toward*; minus sign at the other—the one **P** points *away* from).

(i) $L \gg a$. Then the ends look like point charges, and the whole thing is like a physical dipole, of length L and charge $P\pi a^2$. See Fig. (a).

(ii) $L \ll a$. Then it's like a circular parallel-plate capacitor. Field is nearly uniform inside; nonuniform "fringing field" at the edges. See Fig. (b).

(iii) $L \approx a$. See Fig. (c).



(a) Like a dipole (b) Like a parallel-plate capacitor (c)

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7. *[G 4.31] A dielectric cube of side a, centered at the origin, carries a "frozen-in" polarization* $\mathbf{P} = k\mathbf{r}$*, where k is a constant. Find all the bound charges, and check that they add up to zero.*

$\mathbf{P} = k\mathbf{r} = k(x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + z\,\hat{\mathbf{z}}) \Longrightarrow \rho_b = -\nabla\cdot\mathbf{P} = -k(1+1+1) = \boxed{-3k.}$

Total volume bound charge: $\boxed{Q_{\text{vol}} = -3ka^3.}$

$\sigma_b = \mathbf{P}\cdot\hat{\mathbf{n}}$. At top surface, $\hat{\mathbf{n}} = \hat{\mathbf{z}}$, $z = a/2$; so $\sigma_b = ka/2$. Clearly, $\boxed{\sigma_b = ka/2}$ on all six surfaces.

Total surface bound charge: $\boxed{Q_{\text{surf}} = 6(ka/2)a^2 = 3ka^3.}$ Total bound charge is zero.

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8. *[G 4.19] Suppose you have enough linear dielectric material, of dielectric constant* ϵ_r*, to* half-*fill a parallel-plate capacitor (Fig.). By what fraction is the capacitance increased when you distribute the material as in (a) of given Fig.? How about (b) of the same? For a given potential difference V between the plates, find* **E**, **D**, *and* **P**, *in each region, and the free and bound charge on all surfaces, for both cases.*

With no dielectric, $C_0 = A\epsilon_0/d$

In configuration (a), with $+\sigma$ on upper plate, $-\sigma$ on lower, $D = \sigma$ between the plates. $E = \sigma/\epsilon_0$ (in air) and $E = \sigma/\epsilon$ (in dielectric). So $V = \frac{\sigma}{\epsilon_0}\frac{d}{2} + \frac{\sigma}{\epsilon}\frac{d}{2} = \frac{Qd}{2\epsilon_0 A}\left(1 + \frac{\epsilon_0}{\epsilon}\right)$.

$$C_a = \frac{Q}{V} = \frac{\epsilon_0 A}{d}\left(\frac{2}{1+1/\epsilon_r}\right) \Longrightarrow \boxed{\frac{C_a}{C_0} = \frac{2\epsilon_r}{1+\epsilon_r}.}$$

In configuration (b), with potential difference V: $E = V/d$, so $\sigma = \epsilon_0 E = \epsilon_0 V/d$ (in air).

$P = \epsilon_0\chi_e E = \epsilon_0\chi_e V/d$ (in dielectric), so $\sigma_b = -\epsilon_0\chi_e V/d$ (at top surface of dielectric).

$\sigma_{\text{tot}} = \epsilon_0 V/d = \sigma_f + \sigma_b = \sigma_f - \epsilon_0\chi_e V/d$, so $\sigma_f = \epsilon_0 V(1+\chi_e)/d = \epsilon_0\epsilon_r V/d$ (on top plate above dielectric).

$$\Rightarrow C_b = \frac{Q}{V} = \frac{1}{V}\left(\sigma\frac{A}{2} + \sigma_f\frac{A}{2}\right) = \frac{A}{2V}\left(\epsilon_0\frac{V}{d} + \epsilon_0\frac{V}{d}\epsilon_r\right) = \frac{A\epsilon_0}{d}\left(\frac{1+\epsilon_r}{2}\right). \quad \boxed{\frac{C_b}{C_0} = \frac{1+\epsilon_r}{2}.}$$

[Which is greater? $\frac{C_b}{C_0} - \frac{C_a}{C_0} = \frac{1+\epsilon_r}{2} - \frac{2\epsilon_r}{1+\epsilon_r} = \frac{(1+\epsilon_r)^2 - 4\epsilon_r}{2(1+\epsilon_r)} = \frac{1+2\epsilon_r+4\epsilon_r^2-4\epsilon_r}{2(1+\epsilon_r)} = \frac{(1-\epsilon_r)^2}{2(1+\epsilon_r)} > 0$. So $C_b > C_a$.]

If the x axis points *down*:

	$\mathbf{E}$	$\mathbf{D}$	$\mathbf{P}$	σ_b (top surface)	σ_f (top plate)
(a) air	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	0	0	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{V}{d}$
(a) dielectric	$\frac{2}{(\epsilon_r+1)}\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{2\epsilon_r}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$-\frac{2(\epsilon_r-1)}{(\epsilon_r+1)}\frac{\epsilon_0 V}{d}$	—
(b) air	$\frac{V}{d}\,\hat{\mathbf{x}}$	$\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	0	0	$\frac{\epsilon_0 V}{d}$ (left)
(b) dielectric	$\frac{V}{d}\,\hat{\mathbf{x}}$	$\epsilon_r\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$(\epsilon_r-1)\frac{\epsilon_0 V}{d}\,\hat{\mathbf{x}}$	$-(\epsilon_r-1)\frac{\epsilon_0 V}{d}$	$\epsilon_r\frac{\epsilon_0 V}{d}$ (right)

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9. *[G 4.32] A point charge q is imbedded at the center of a sphere of linear dielectric material (with susceptibility χ_e and radius $\mathbf{R}$). Find the electric field, the polarization, and the bound charge densities, ρ_b and σ_b. What is the total bound charge on the surface? Where is the compensating negative bound charge located?*

$$\oint \mathbf{D}\cdot d\mathbf{a} = Q_{f_{enc}} \Rightarrow \mathbf{D} = \frac{q}{4\pi r^2}\hat{\mathbf{r}};\ \mathbf{E} = \frac{1}{\epsilon}\mathbf{D} = \boxed{\frac{q}{4\pi\epsilon_0(1+\chi_e)}\frac{\hat{\mathbf{r}}}{r^2}};\ \mathbf{P} = \epsilon_0\chi_e\mathbf{E} = \boxed{\frac{q\chi_e}{4\pi(1+\chi_e)}\frac{\hat{\mathbf{r}}}{r^2}}.$$

$$\rho_b = -\nabla\cdot\mathbf{P} = -\frac{q\chi_e}{4\pi(1+\chi_e)}\left(\nabla\cdot\frac{\hat{\mathbf{r}}}{r^2}\right) = \boxed{-q\frac{\chi_e}{1+\chi_e}\delta^3(\mathbf{r})}\ \text{(Eq. 1.99)};\ \sigma_b = \mathbf{P}\cdot\hat{\mathbf{r}} = \boxed{\frac{q\chi_e}{4\pi(1+\chi_e)R^2}};$$

$Q_{\text{surf}} = \sigma_b(4\pi R^2) = \boxed{q\frac{\chi_e}{1+\chi_e}}.$ The compensating negative charge is at the center:

$$\int \rho_b\,d\tau = -\frac{q\chi_e}{1+\chi_e}\int \delta^3(\mathbf{r})\,d\tau = -q\frac{\chi_e}{1+\chi_e}.$$

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10. *[G 4.36] A conducting sphere at potential V_0 is half embedded in linear dielectric material of susceptibility χ_e, which occupies the region $z < 0$ shown in the first Fig. Claim: the potential everywhere is exactly the same as it would have been in the absence of the dielectric! Check the claim as follows:*

   (a) *Write down the formula for the proposed potential $V(r)$, in terms of V_0, $\mathbf{R}$, and r. Use it to determine the field, the polarization, the bound charge, and the free charge distribution on the sphere.*

(b) *Show that the total charge configuration would indeed produce the potential $V(r)$.*

(c) *Appeal to the uniqueness theorem to complete the argument.*

(d) *Could you solve the configurations in the second Fig. with the same potential? If not, explain why.*

(a) Proposed potential: $\boxed{V(r) = V_0 \frac{R}{r}.}$ If so, then $\boxed{\mathbf{E} = -\nabla V = V_0 \frac{R}{r^2}\,\hat{\mathbf{r}},}$ in which case $\boxed{\mathbf{P} = \epsilon_0 \chi_e V_0 \frac{R}{r^2}\,\hat{\mathbf{r}},}$ in the region $z < 0$. ($\mathbf{P} = 0$ for $z > 0$, of course.) Then $\sigma_b = \epsilon_0 \chi_e V_0 \frac{R}{R^2}(\hat{\mathbf{r}}\cdot\hat{\mathbf{n}}) = \boxed{-\frac{\epsilon_0 \chi_e V_0}{R}.}$ (*Note:* $\hat{\mathbf{n}}$ points *out* of dielectric $\Rightarrow \hat{\mathbf{n}} = -\hat{\mathbf{r}}$.) This σ_b is on the surface at $r = R$. The flat surface $z = 0$ carries no bound charge, since $\hat{\mathbf{n}} = \hat{\mathbf{z}} \perp \hat{\mathbf{r}}$. Nor is there any volume bound charge (Eq. 4.39). If V is to have the required spherical symmetry, the *net* charge must be uniform:

$\sigma_{tot} 4\pi R^2 = Q_{tot} = 4\pi\epsilon_0 R V_0$ (since $V_0 = Q_{tot}/4\pi\epsilon_0 R$), so $\sigma_{tot} = \epsilon_0 V_0/R$. Therefore

$$\boxed{\sigma_f = \begin{Bmatrix} (\epsilon_0 V_0/R), & \text{on northern hemisphere} \\ (\epsilon_0 V_0/R)(1+\chi_e), & \text{on southern hemisphere} \end{Bmatrix}.}$$

(b) By construction, $\sigma_{tot} = \sigma_b + \sigma_f = \epsilon_0 V_0/R$ is uniform (on the northern hemisphere $\sigma_b = 0$, $\sigma_f = \epsilon_0 V_0/R$; on the southern hemisphere $\sigma_b = -\epsilon_0 \chi_e V_0/R$, so $\sigma_f = \epsilon V_0/R$). The potential of a uniformly charged sphere is

$$V_0 = \frac{Q_{tot}}{4\pi\epsilon_0 r} = \frac{\sigma_{tot}(4\pi R^2)}{4\pi\epsilon_0 r} = \frac{\epsilon_0 V_0}{R}\frac{R^2}{\epsilon_0 r} = V_0 \frac{R}{r}.$$

(c) Since everything is consistent, and the boundary conditions ($V = V_0$ at $r = R$, $V \to 0$ at ∞) are met, Prob. 4.35 guarantees that this is *the* solution.

(d) Figure (b) works the same way, but Fig. (a) does *not:* on the flat surface, $\mathbf{P}$ is *not* perpendicular to $\hat{\mathbf{n}}$, so we'd get bound charge on this surface, spoiling the symmetry.