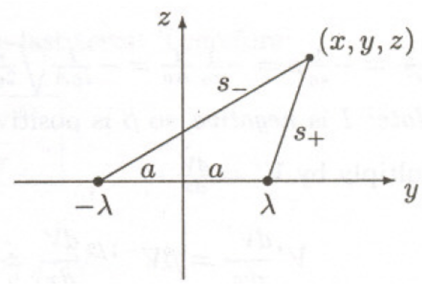
1. *[G 3.11] Two long, straight copper pipes, each of radius R, are held a distance 2d apart. One is at potential V_0, the other at $-V_0$. Find the potential everywhere.*

Problem 2.47

(a) Potential of $+\lambda$ is $V_+ = -\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_+}{a}\right)$, where s_+ is distance from λ_+ (Prob. 2.22).
Potential of $-\lambda$ is $V_- = +\frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{a}\right)$, where s_- is distance from λ_-.

$\therefore$ Total $\boxed{V = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{s_-}{s_+}\right).}$



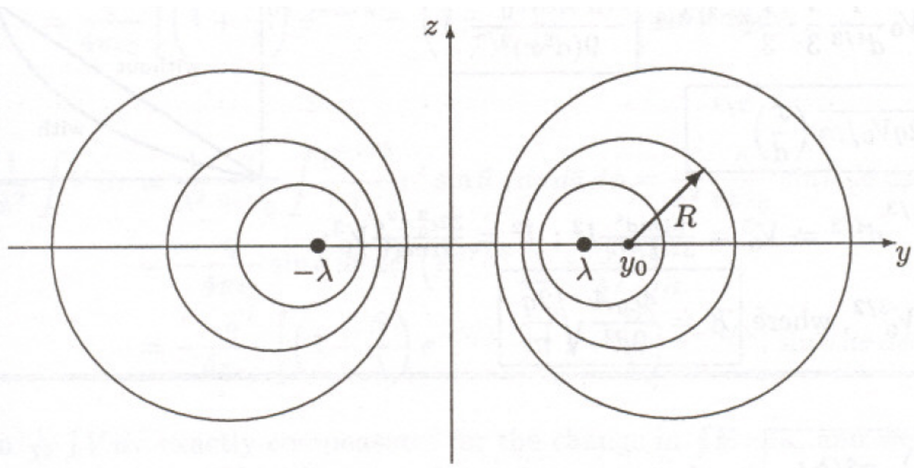
Now $s_+ = \sqrt{(y-a)^2+z^2}$, and $s_- = \sqrt{(y+a)^2+z^2}$, so

$$V(x,y,z) = \frac{\lambda}{2\pi\epsilon_0}\ln\left(\frac{\sqrt{(y+a)^2+z^2}}{\sqrt{(y-a)^2+z^2}}\right) = \boxed{\frac{\lambda}{4\pi\epsilon_0}\ln\left[\frac{(y+a)^2+z^2}{(y-a)^2+z^2}\right]}.$$

(b) Equipotentials are given by $\frac{(y+a)^2+z^2}{(y-a)^2+z^2} = e^{(4\pi\epsilon_0 V_0/\lambda)} = k =$ constant. That is:
$y^2+2ay+a^2+z^2 = k(y^2-2ay+a^2+z^2) \Rightarrow y^2(k-1)+z^2(k-1)+a^2(k-1)-2ay(k+1)=0$, or
$y^2+z^2+a^2-2ay\left(\frac{k+1}{k-1}\right)=0$. The equation for a *circle*, with center at $(y_0, 0)$ and radius R, is
$(y-y_0)^2+z^2=R^2$, or $y^2+z^2+(y_0^2-R^2)-2yy_0=0$.
Evidently the equipotentials *are* circles, with $y_0 = a\left(\frac{k+1}{k-1}\right)$ and
$a^2 = y_0^2 - R^2 \Rightarrow R^2 = y_0^2 - a^2 = a^2\left(\frac{k+1}{k-1}\right)^2 - a^2 = a^2\frac{(k^2+2k+1-k^2+2k-1)}{(k-1)^2} = a^2\frac{4k}{(k-1)^2}$, or
$R = \frac{2a\sqrt{k}}{|k-1|}$; or, in terms of V_0:

$$y_0 = a\frac{e^{4\pi\epsilon_0 V_0/\lambda}+1}{e^{4\pi\epsilon_0 V_0/\lambda}-1} = a\frac{e^{2\pi\epsilon_0 V_0/\lambda}+e^{-2\pi\epsilon_0 V_0/\lambda}}{e^{2\pi\epsilon_0 V_0/\lambda}-e^{-2\pi\epsilon_0 V_0/\lambda}} = \boxed{a\coth\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)}.$$

$$R = 2a\frac{e^{2\pi\epsilon_0 V_0/\lambda}}{e^{4\pi\epsilon_0 V_0/\lambda}-1} = a\frac{2}{(e^{2\pi\epsilon_0 V_0/\lambda}-e^{-2\pi\epsilon_0 V_0/\lambda})} = \frac{a}{\sinh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)} = \boxed{a\,\mathrm{csch}\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right)}.$$



From Prob. 2.47 (with $y_0 \to d$): $V = \dfrac{\lambda}{4\pi\epsilon_0} \ln\left[\dfrac{(x+a)^2+y^2}{(x-a)^2+y^2}\right]$, where $a^2 = y_0{}^2 - R^2 \Rightarrow$ $a = \sqrt{d^2 - R^2}$,

and

$$\left\{\begin{array}{l} a\coth(2\pi\epsilon_0 V_0/\lambda) = d \\ a\,\text{csch}(2\pi\epsilon_0 V_0/\lambda) = R \end{array}\right\} \Rightarrow \text{(dividing)} \quad \frac{d}{R} = \cosh\left(\frac{2\pi\epsilon_0 V_0}{\lambda}\right), \text{ or } \quad \lambda = \frac{2\pi\epsilon_0 V_0}{\cosh^{-1}(d/R)}.$$

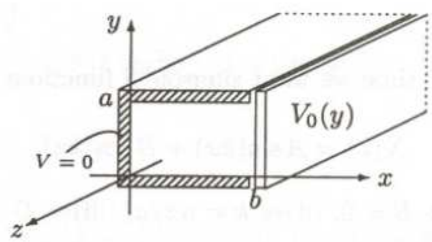
---

2. *[G 3.14] A rectangular pipe, running parallel to the $z-$axis (from $-\infty$ to ∞), has three grounded metal sides, at $y=0$, $y=a$, and $x=0$. The fourth side, at $x=b$, is maintained at a specific potential $V_0(y)$.*

   (a) *Develop a general formula for the potential within the pipe.*

   (b) *Find the potential explicitly, for the case $V_0(y) = V_0$ (a constant).*

(a) $\dfrac{\partial^2 V}{\partial x^2} + \dfrac{\partial^2 V}{\partial y^2} = 0$, with boundary conditions



$$\left\{\begin{array}{ll} \text{(i)} & V(x,0) = 0, \\ \text{(ii)} & V(x,a) = 0, \\ \text{(iii)} & V(0,y) = 0, \\ \text{(iv)} & V(b,y) = V_0(y). \end{array}\right\}$$

As in Ex. 3.4, separation of variables yields

$$V(x,y) = \left(Ae^{kx} + Be^{-kx}\right)(C\sin ky + D\cos ky).$$

Here (i)$\Rightarrow D = 0$, (iii)$\Rightarrow B = -A$, (ii)$\Rightarrow ka$ is an integer multiple of π:

$$V(x,y) = AC\left(e^{n\pi x/a} - e^{-n\pi x/a}\right)\sin(n\pi y/a) = (2AC)\sinh(n\pi x/a)\sin(n\pi y/a).$$

But $(2AC)$ is a constant, and the most general linear combination of separable solutions consistent with (i), (ii), (iii) is

$$V(x,y) = \sum_{n=1}^{\infty} C_n \sinh(n\pi x/a)\sin(n\pi y/a).$$

It remains to determine the coefficients C_n so as to fit boundary condition (iv):

$$\sum C_n \sinh(n\pi b/a)\sin(n\pi y/a) = V_0(y). \text{ Fourier's trick} \Rightarrow C_n \sinh(n\pi b/a) = \frac{2}{a}\int^{a} V_0(y)\sin(n\pi y/a)\,dy.$$

Therefore

$$C_n = \frac{2}{a\sinh(n\pi b/a)}\int_0^a V_0(y)\sin(n\pi y/a)\,dy.$$

(b) $C_n = \frac{2}{a\sinh(n\pi b/a)} V_0 \int_0^a \sin(n\pi y/a)\,dy = \frac{2V_0}{a\sinh(n\pi b/a)} \times \left\{ \begin{array}{ll} 0, & \text{if } n \text{ is even,} \\ \frac{2a}{n\pi}, & \text{if } n \text{ is odd.} \end{array} \right\}$

$$\boxed{V(x,y) = \frac{4V_0}{\pi} \sum_{n=1,3,5,\ldots} \frac{\sinh(n\pi x/a)\sin(n\pi y/a)}{n\sinh(n\pi b/a)}}.$$

---

3. *[G 3.1] Find the average potential over a spherical surface of radius R due to a point charge q located* inside. *Show that in general,*
$V_{ave} = V_{center} + \frac{Q_{enc}}{4\pi\epsilon_0 R}$,
where V_{center} is the potential at the center due to all the external *charges, and Q_{enc} is the total enclosed charge.*

The argument is exactly the same as in Sect. 3.1.4, except that since $z < R$, $\sqrt{z^2 + R^2 - 2zR} = (R - z)$, instead of $(z - R)$. Hence $V_{\text{ave}} = \frac{q}{4\pi\epsilon_0}\frac{1}{2zR}[(z+R) - (R-z)] = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{R}}$. If there is more than one charge inside the sphere, the average potential due to interior charges is $\frac{1}{4\pi\epsilon_0}\frac{Q_{\text{enc}}}{R}$, and the average due to exterior charges is V_{center}, so $V_{\text{ave}} = V_{\text{center}} + \frac{Q_{\text{enc}}}{4\pi\epsilon_0 R}$. ✓

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4. *[G 3.3] Find the general solution to Laplace's equation in spherical coordinates, for the case where V depends only on r. Do the same for cylindrical coordinates, assuming V depends only on s.*

Laplace's equation in *spherical* coordinates, for V dependent only on r, reads:

$$\nabla^2 V = \frac{1}{r^2}\frac{d}{dr}\left(r^2 \frac{dV}{dr}\right) = 0 \Rightarrow r^2\frac{dV}{dr} = c \text{ (constant)} \Rightarrow \frac{dV}{dr} = \frac{c}{r^2} \Rightarrow \boxed{V = -\frac{c}{r} + k.}$$

Example: potential of a uniformly charged sphere.

In *cylindrical* coordinates: $\nabla^2 V = \frac{1}{s}\frac{d}{ds}\left(s\frac{dV}{ds}\right) = 0 \Rightarrow s\frac{dV}{ds} = c \Rightarrow \frac{dV}{ds} = \frac{c}{s} \Rightarrow \boxed{V = c\ln s + k.}$

Example: potential of a long wire.

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5. *[G 3.7]*
(a) Using the law of cosines, show that $V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{r} + \frac{q'}{r'}\right)$ *(where r and r′ are the distances from q and q′ respectively) can be written as follows:*
$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left[\frac{q}{\sqrt{r^2+s^2-2rs\cos\theta}} - \frac{q}{\sqrt{R^2+(rs/R)^2-2rs\cos\theta}}\right]$,
where r and θ are the usual spherical polar coordinates, with the z- axis along the line through q. In this form it is obvious that V = 0 on the sphere, r = R.
(b) Find the induced surface charge on the sphere, as a function of θ. Integrate this to get the total induced charge.
(c) Calculate the energy of this configuration.

(a) From Fig. 3.13: $\mathscr{r} = \sqrt{r^2 + a^2 - 2ra\cos\theta}$; $\mathscr{r}' = \sqrt{r^2 + b^2 - 2rb\cos\theta}$. Therefore:

$$\frac{q'}{\mathscr{r}'} = -\frac{R}{a}\frac{q}{\sqrt{r^2 + b^2 - 2rb\cos\theta}} \quad \text{(Eq. 3.15), while } b = \frac{R^2}{a} \text{ (Eq. 3.16).}$$

$$= -\frac{q}{\left(\frac{a}{R}\right)\sqrt{r^2 + \frac{R^4}{a^2} - 2r\frac{R^2}{a}\cos\theta}} = -\frac{q}{\sqrt{\left(\frac{ar}{R}\right)^2 + R^2 - 2ra\cos\theta}}.$$

Therefore:

$$V(r,\theta) = \frac{1}{4\pi\epsilon_0}\left(\frac{q}{\mathscr{r}} + \frac{q'}{\mathscr{r}'}\right) = \boxed{\frac{q}{4\pi\epsilon_0}\left\{\frac{1}{\sqrt{r^2 + a^2 - 2ra\cos\theta}} - \frac{1}{\sqrt{R^2 + (ra/R)^2 - 2ra\cos\theta}}\right\}}.$$

Clearly, when $r = R$, $V \to 0$.

(b) $\sigma = -\epsilon_0 \frac{\partial V}{\partial n}$ (Eq. 2.49). In this case, $\frac{\partial V}{\partial n} = \frac{\partial V}{\partial r}$ at the point $r = R$. Therefore,

$$\sigma(\theta) = -\epsilon_0\left(\frac{q}{4\pi\epsilon_0}\right)\left\{-\frac{1}{2}(r^2 + a^2 - 2ra\cos\theta)^{-3/2}(2r - 2a\cos\theta) + \frac{1}{2}\left(R^2 + (ra/R)^2 - 2ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R^2}2r - 2a\cos\theta\right)\right\}\Bigg|_{r=R}$$

$$= -\frac{q}{4\pi}\left\{-(R^2 + a^2 - 2Ra\cos\theta)^{-3/2}(R - a\cos\theta) + \left(R^2 + a^2 - 2Ra\cos\theta\right)^{-3/2}\left(\frac{a^2}{R} - a\cos\theta\right)\right\}$$

$$= \frac{q}{4\pi}(R^2 + a^2 - 2Ra\cos\theta)^{-3/2}\left[R - a\cos\theta - \frac{a^2}{R} + a\cos\theta\right]$$

$$= \boxed{\frac{q}{4\pi R}(R^2 - a^2)(R^2 + a^2 - 2Ra\cos\theta)^{-3/2}.}$$

$$q_{\text{induced}} = \int \sigma\, da = \frac{q}{4\pi R}(R^2 - a^2)\int (R^2 + a^2 - 2Ra\cos\theta)^{-3/2}R^2\sin\theta\, d\theta\, d\phi$$

$$= \frac{q}{4\pi R}(R^2 - a^2)2\pi R^2\left[-\frac{1}{Ra}(R^2 + a^2 - 2Ra\cos\theta)^{-1/2}\right]\Bigg|_0^\pi$$

$$= \frac{q}{2a}(a^2 - R^2)\left[\frac{1}{\sqrt{R^2 + a^2 + 2Ra}} - \frac{1}{\sqrt{R^2 + a^2 - 2Ra}}\right].$$

But $a > R$ (else q would be *inside*), so $\sqrt{R^2 + a^2 - 2Ra} = a - R$.

$$= \frac{q}{2a}(a^2 - R^2)\left[\frac{1}{(a+R)} - \frac{1}{(a-R)}\right] = \frac{q}{2a}\left[(a - R) - (a + R)\right] = \frac{q}{2a}(-2R)$$

$$= \boxed{-\frac{qR}{a} = q'.}$$

(c) The force on q, due to the sphere, is the same as the force of the image charge q', to wit:

$$F = \frac{1}{4\pi\epsilon_0}\frac{qq'}{(a-b)^2} = \frac{1}{4\pi\epsilon_0}\left(-\frac{R}{a}q^2\right)\frac{1}{(a-R^2/a)^2} = -\frac{1}{4\pi\epsilon_0}\frac{q^2Ra}{(a^2-R^2)^2}.$$

To bring q in from infinity to a, then, we do work

$$W = \frac{q^2R}{4\pi\epsilon_0}\int_\infty^a \frac{\bar{a}}{(\bar{a}^2-R^2)^2}\,d\bar{a} = \frac{q^2R}{4\pi\epsilon_0}\left[-\frac{1}{2}\frac{1}{(\bar{a}^2-R^2)}\right]\Bigg|_\infty^a = \boxed{-\frac{1}{4\pi\epsilon_0}\frac{q^2R}{2(a^2-R^2)}}.$$

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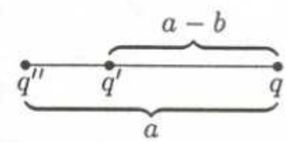
6. *[G 3.8] Consider a point charge q situated at a distance a from the center of a grounded conducting sphere of radius R. The same basic model will handle the case of a sphere at* any *potential V_0 (relative to infinity) with the addition of a second image charge. What charge should you use, and where should you put it? Find the force of attraction between a point charge q and a* neutral *conducting sphere.*

Place a second image charge, q'', at the *center* of the sphere; this will not alter the fact that the sphere is an *equipotential*, but merely *increase* that potential from zero to $V_0 = \frac{1}{4\pi\epsilon_0}\frac{q''}{R}$;



$\boxed{q'' = 4\pi\epsilon_0 V_0 R \text{ at center of sphere.}}$

For a *neutral* sphere, $q' + q'' = 0$.

$$\begin{aligned}
F &= \frac{1}{4\pi\epsilon_0}q\left(\frac{q''}{a^2}+\frac{q'}{(a-b)^2}\right) = \frac{qq'}{4\pi\epsilon_0}\left(-\frac{1}{a^2}+\frac{1}{(a-b)^2}\right) \\
&= \frac{qq'}{4\pi\epsilon_0}\frac{b(2a-b)}{a^2(a-b)^2} = \frac{q(-Rq/a)}{4\pi\epsilon_0}\frac{(R^2/a)(2a-R^2/a)}{a^2(a-R^2/a)^2} \\
&= \boxed{-\frac{q^2}{4\pi\epsilon_0}\left(\frac{R}{a}\right)^3\frac{(2a^2-R^2)}{(a^2-R^2)^2}}.
\end{aligned}$$

(Drop the minus sign, because the problem asks for the force of *attraction*.)

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7. *[G 3.12] Two infinite grounded metal plates lie parallel to the xz plane, one at $y=0$, the other at $y=a$. The left end, at $x=0$, consists of two metal strips: one, from $y=0$ to $y=a/2$, is held at a constant potential V_0, and the other, from $y=a/2$ to $y=a$, is at potential $-V_0$. Find the potential in the infinite slot.*

$$V(x,y) = \sum_{n=1}^{\infty} C_n e^{-n\pi x/a}\sin(n\pi y/a) \quad \text{(Eq. 3.30)}, \quad \text{where} \quad C_n = \frac{2}{a}\int_0^a V_0(y)\sin(n\pi y/a)\,dy \quad \text{(Eq. 3.34)}.$$

In this case $V_0(y) = \left\{\begin{array}{ll} +V_0, & \text{for } 0<y<a/2 \\ -V_0, & \text{for } a/2<y<a \end{array}\right\}$. Therefore,

$$\begin{aligned}
C_n &= \frac{2}{a}V_0\left\{\int_0^{a/2}\sin(n\pi y/a)\,dy - \int_{a/2}^{a}\sin(n\pi y/a)\,dy\right\} = \frac{2V_0}{a}\left\{-\frac{\cos(n\pi y/a)}{(n\pi/a)}\Bigg|_0^{a/2} + \frac{\cos(n\pi y/a)}{(n\pi/a)}\Bigg|_{a/2}^{a}\right\} \\
&= \frac{2V_0}{n\pi}\left\{-\cos\left(\frac{n\pi}{2}\right)+\cos(0)+\cos(n\pi)-\cos\left(\frac{n\pi}{2}\right)\right\} = \frac{2V_0}{n\pi}\left\{1+(-1)^n-2\cos\left(\frac{n\pi}{2}\right)\right\}.
\end{aligned}$$

The term in curly brackets is:

$$\left\{\begin{array}{lll} n=1 & : & 1-1-2\cos(\pi/2)=0, \\ n=2 & : & 1+1-2\cos(\pi)=4, \\ n=3 & : & 1-1-2\cos(3\pi/2)=0, \\ n=4 & : & 1+1-2\cos(2\pi)=0, \end{array}\right\} \text{etc. (Zero if } n \text{ is odd or divisible by 4, otherwise 4.)}$$

Therefore

$$C_n = \begin{cases} 8V_0/n\pi, & n = 2, 6, 10, 14, \text{etc. (in general, } 4j+2, \text{ for } j = 0, 1, 2, ...), \\ 0, & \text{otherwise.} \end{cases}$$

So

$$\boxed{V(x,y) = \frac{8V_0}{\pi} \sum_{n=2,6,10,...} \frac{e^{-n\pi x/a}\sin(n\pi y/a)}{n}} = \frac{8V_0}{\pi} \sum_{j=0}^{\infty} \frac{e^{-(4j+2)\pi x/a}\sin[(4j+2)\pi y/a]}{(4j+2)}.$$

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8. *[G 3.15] A cubical box (sides of length a) consists of five metal plates, which are welded together and grounded. The top is made of a separate sheet of metal, insulated from the others, and held at a constant potential V_0. Find the potential inside the box.*

Same format as Ex. 3.5, only the boundary conditions are:

$$\left\{\begin{array}{llll} \text{(i)} & V=0 & \text{when} & x=0, \\ \text{(ii)} & V=0 & \text{when} & x=a, \\ \text{(iii)} & V=0 & \text{when} & y=0, \\ \text{(iv)} & V=0 & \text{when} & y=a, \\ \text{(v)} & V=0 & \text{when} & z=0, \\ \text{(vi)} & V=V_0 & \text{when} & z=a. \end{array}\right\}$$

This time we want sinusoidal functions in x and y, exponential in z:

$$X(x) = A\sin(kx) + B\cos(kx), \quad Y(y) = C\sin(ly) + D\cos(ly), \quad Z(z) = Ee^{\sqrt{k^2+l^2}z} + Ge^{-\sqrt{k^2+l^2}z}.$$

(i)$\Rightarrow$ $B = 0$; (ii)$\Rightarrow$ $k = n\pi/a$; (iii)$\Rightarrow$ $D = 0$; (iv)$\Rightarrow$ $l = m\pi/a$; (v)$\Rightarrow$ $E + G = 0$. Therefore

$$Z(z) = 2E\sinh(\pi\sqrt{n^2+m^2}z/a).$$

Putting this all together, and combining the constants, we have:

$$V(x,y,z) = \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} C_{n,m}\sin(n\pi x/a)\sin(m\pi y/a)\sinh(\pi\sqrt{n^2+m^2}z/a).$$

It remains to evaluate the constants $C_{n,m}$, by imposing boundary condition (vi):

$$V_0 = \sum\sum\left[C_{n,m}\sinh(\pi\sqrt{n^2+m^2})\right]\sin(n\pi x/a)\sin(m\pi y/a).$$

According to Eqs. 3.50 and 3.51:

$$C_{n,m}\sinh\left(\pi\sqrt{n^2+m^2}\right) = \left(\frac{2}{a}\right)^2 V_0 \int_0^a\int_0^a \sin(n\pi x/a)\sin(m\pi y/a)\,dx\,dy = \begin{Bmatrix} 0, & \text{if } n \text{ or } m \text{ is even,} \\ \dfrac{16V_0}{\pi^2 nm}, & \text{if both are odd.} \end{Bmatrix}$$

Therefore

$$\boxed{V(x,y,z) = \frac{16V_0}{\pi^2}\sum_{n=1,3,5,\dots}\sum_{m=1,3,5,\dots}\frac{1}{nm}\sin(n\pi x/a)\sin(m\pi y/a)\frac{\sinh\left(\pi\sqrt{n^2+m^2}\,z/a\right)}{\sinh\left(\pi\sqrt{n^2+m^2}\right)}.}$$
