[Note: Only the first five problems will be discussed in tutorial classes. You should attempt the remaining problems and in case you get stuck, ask your tutorial instructor]

1. *[G 2.34] Consider two concentric spherical shells, of radii a and b. Suppose the inner one carries a charge q, and the outer one a charge −q (both of them uniformly distributed over the surface). Calculate the energy of this configuration, (a) using* $W = \frac{\epsilon_0}{2}\int E^2 d\tau$*, and (b) using* $W_{tot} = W_1 + W_2 + \epsilon_0 \int \mathbf{E}_1 \cdot \mathbf{E}_2 d\tau$.

   (a) $W = \frac{\epsilon_0}{2}\int E^2 d\tau$. $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$ $(a < r < b)$, zero elsewhere.
   $W = \frac{\epsilon_0}{2}\left(\frac{q}{4\pi\epsilon_0}\right)^2 \int_a^b \left(\frac{1}{r^2}\right)^2 4\pi r^2 dr = \frac{q^2}{8\pi\epsilon_0}\int_a^b \frac{1}{r^2} dr = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right).$

   (b) $W_1 = \frac{1}{8\pi\epsilon_0}\frac{q^2}{a}$, $W_2 = \frac{1}{8\pi\epsilon_0}\frac{q^2}{b}$, $\mathbf{E_1} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}\ (r > a)$, $\mathbf{E_2} = \frac{1}{4\pi\epsilon_0}\frac{-q}{r^2}\hat{\mathbf{r}}\ (r > b)$. So
   $\mathbf{E_1}\cdot\mathbf{E_2} = \left(\frac{1}{4\pi\epsilon_0}\right)^2 \frac{-q^2}{r^4}$, $(r > b)$, and hence $\int \mathbf{E_1}\cdot\mathbf{E_2} d\tau = -\left(\frac{1}{4\pi\epsilon_0}\right)^2 q^2 \int_b^\infty \frac{1}{r^4} 4\pi r^2 dr = -\frac{q^2}{4\pi\epsilon_0 b}$.
   $W_{tot} = W_1 + W_2 + \epsilon_0 \int \mathbf{E_1}\cdot\mathbf{E_2} d\tau = \frac{1}{8\pi\epsilon_0} q^2 \left(\frac{1}{a} + \frac{1}{b} - \frac{2}{b}\right) = \frac{q^2}{8\pi\epsilon_0}\left(\frac{1}{a} - \frac{1}{b}\right).$

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2. *[G 2.45] A sphere of radius R carries a charge density* $\rho(r) = kr$ *(where k is a constant). Find the energy of the configuration. Check your answer by calculating it in at least two different ways.*

First let's determine the electric field inside and outside the sphere, using Gauss's law:

$$\epsilon_0 \oint \mathbf{E}\cdot d\mathbf{a} = \epsilon_0 4\pi r^2 E = Q_{enc} = \int \rho\, d\tau = \int (k\bar{r})\bar{r}^2 \sin\theta\, d\bar{r}\, d\theta\, d\phi = 4\pi k \int_0^r \bar{r}^3 d\bar{r} = \begin{cases} \pi k r^4 & (r < R), \\ \pi k R^4 & (r > R). \end{cases}$$

So $\mathbf{E} = \frac{k}{4\epsilon_0} r^2 \hat{\mathbf{r}}\ (r < R)$; $\mathbf{E} = \frac{kR^4}{4\epsilon_0 r^2}\hat{\mathbf{r}}\ (r > R)$.

Method I:

$$W = \frac{\epsilon_0}{2}\int E^2 d\tau \text{ (Eq. 2.45)} = \frac{\epsilon_0}{2}\int_0^R \left(\frac{kr^2}{4\epsilon_0}\right)^2 4\pi r^2 dr + \frac{\epsilon_0}{2}\int_R^\infty \left(\frac{kR^4}{4\epsilon_0 r^2}\right)^2 4\pi r^2 dr$$
$$= 4\pi\frac{\epsilon_0}{2}\left(\frac{k}{4\epsilon_0}\right)^2 \left\{\int_0^R r^6 dr + R^8 \int_R^\infty \frac{1}{r^2} dr\right\} = \frac{\pi k^2}{8\epsilon_0}\left\{\frac{R^7}{7} + R^8\left(-\frac{1}{r}\right)\Big|_R^\infty\right\} = \frac{\pi k^2}{8\epsilon_0}\left(\frac{R^7}{7} + R^7\right)$$
$$= \boxed{\frac{\pi k^2 R^7}{7\epsilon_0}}.$$

Method II:

$$W = \frac{1}{2}\int \rho V\, d\tau \quad \text{(Eq. 2.43)}.$$

For $r < R$, $$V(r) = -\int_\infty^r \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^R \left(\frac{kR^4}{4\epsilon_0 r^2}\right) dr - \int_R^r \left(\frac{kr^2}{4\epsilon_0}\right) dr = -\frac{k}{4\epsilon_0}\left\{R^4\left(-\frac{1}{r}\right)\Big|_\infty^R + \frac{r^3}{3}\Big|_R^r\right\}$$
$$= -\frac{k}{4\epsilon_0}\left(-R^3 + \frac{r^3}{3} - \frac{R^3}{3}\right) = \frac{k}{3\epsilon_0}\left(R^3 - \frac{r^3}{4}\right).$$
$$\therefore W = \frac{1}{2}\int_0^R (kr)\left[\frac{k}{3\epsilon_0}\left(R^3 - \frac{r^3}{4}\right)\right] 4\pi r^2 dr = \frac{2\pi k^2}{3\epsilon_0}\int_0^R \left(R^3 r^3 - \frac{1}{4}r^6\right) dr$$
$$= \frac{2\pi k^2}{3\epsilon_0}\left\{R^3\frac{R^4}{4} - \frac{1}{4}\frac{R^7}{7}\right\} = \frac{\pi k^2 R^7}{2\cdot 3\epsilon_0}\left(\frac{6}{7}\right) = \frac{\pi k^2 R^7}{7\epsilon_0}.$$

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3. *[G 2.36] Two spherical cavities, of radii a and b, are hollowed out from the interior of a (neutral) conducting sphere of radius R (Fig.). At the center of each cavity a point charge is placed - call these charges q_a and q_b.*

   (a) Find the surface charges σ_a, σ_b, and σ_R.

   (b) What is the field outside the conductor?

   (c) What is the field within each cavity?

   (d) What is the force on $q_a$and q_b?

   (e) Which of these answers would change if a third charge, q_c, were brought near the conductor?

(a) $\boxed{\sigma_a = -\frac{q_a}{4\pi a^2}}$; $\boxed{\sigma_b = -\frac{q_b}{4\pi b^2}}$; $\boxed{\sigma_R = \frac{q_a + q_b}{4\pi R^2}}$.

(b) $\boxed{\mathbf{E}_{\text{out}} = \frac{1}{4\pi\epsilon_0}\frac{q_a + q_b}{r^2}\hat{\mathbf{r}},}$ where $\mathbf{r}$ = vector from center of large sphere.

(c) $\boxed{\mathbf{E}_a = \frac{1}{4\pi\epsilon_0}\frac{q_a}{r_a^2}\hat{\mathbf{r}}_a, \quad \mathbf{E}_b = \frac{1}{4\pi\epsilon_0}\frac{q_b}{r_b^2}\hat{\mathbf{r}}_b,}$ where $\mathbf{r}_a$ ($\mathbf{r}_b$) is the vector from center of cavity a (b).

(d) $\boxed{\text{Zero.}}$

(e) σ_R changes (but not σ_a or σ_b); $\mathbf{E}_{\text{outside}}$ changes (but not $\mathbf{E}_a$ or $\mathbf{E}_b$); force on q_a and q_b still zero.

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4. *[G 2.43] Find the net force that the southern hemisphere of a uniformly charged sphere exerts on the northern hemisphere. Express your answer in terms of the radius R and the total charge Q.*

From Prob. 2.12, the field inside a uniformly charged sphere is: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\mathbf{r}$. So the force per unit volume is $\mathbf{f} = \rho\mathbf{E} = \left(\frac{Q}{\frac{4}{3}\pi R^3}\right)\left(\frac{Q}{4\pi\epsilon_0 R^3}\right)\mathbf{r} = \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2\mathbf{r}$, and the force in the z direction on $d\tau$ is:

$$dF_z = f_z\,d\tau = \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 r\cos\theta(r^2\sin\theta\,dr\,d\theta\,d\phi).$$

The total force on the "northern" hemisphere is:

$$F_z = \int f_z\,d\tau = \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \int_0^R r^3\,dr \int_0^{\pi/2}\cos\theta\sin\theta\,d\theta \int_0^{2\pi} d\phi$$
$$= \frac{3}{\epsilon_0}\left(\frac{Q}{4\pi R^3}\right)^2 \left(\frac{R^4}{4}\right)\left(\left.\frac{\sin^2\theta}{2}\right|_0^{\pi/2}\right)(2\pi) = \boxed{\frac{3Q^2}{64\pi\epsilon_0 R^2}}.$$

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5. *Let q_1 (> 0) and $-q_2$, (where $q_2 > 0$) be two charges located at $(d,0,0)$ and $(-d,0,0)$ respectively. Show that the zero potential surface is spherical with center on x axis. If center of the surface is at $(c,0,0)$ and radius is R, find c/d and R/d in terms of q_1/q_2. Sketch surfaces for $q_1/q_2 = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2 3 4$.*

If a point (x,y,z) is on $V = 0$ surface, then

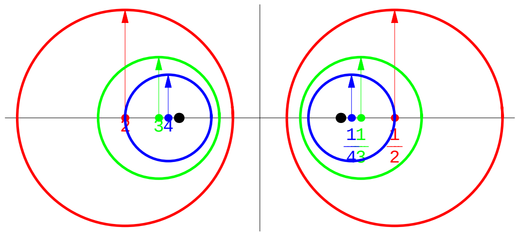
$$\frac{q_1}{\left((x-d)^2 + y^2 + z^2\right)} = \frac{q_2}{\left((x+d)^2 + y^2 + z^2\right)}$$

$$\therefore \quad \left[x - d\frac{1+\alpha^2}{1-\alpha^2}\right]^2 + y^2 + z^2 = d^2\frac{4\alpha^2}{|1-\alpha^2|^2}$$

where, $\alpha = q_1/q_2$. This is an equation of a sphere with centre

$$c = d\frac{1+\alpha^2}{1-\alpha^2} \quad \text{and radius} \quad R = d\frac{2\alpha}{|1-\alpha^2|}$$

Figure shows surfaces for $\alpha = \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, 1, 2\,3\,4$. As the ratio $\alpha \to 1$, $R \to \infty$ and $c \to \pm\infty$ and equipotential surface approaches yz plane.



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6. **[G 2.35]** *A metal sphere of radius R, carrying charge q, is surrounded by a thick concentric metal shell (inner radius a, outer radius b, as shown in Fig.). The shell carries no net charge.*

   (a) *Find the surface charge density σ at R, at a, and at b.*

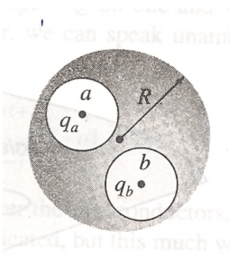
   (b) *Find the potential at the center, using infinity as the reference point.*

   (c) *Now the outer surface is touched to a grounding wire, which lowers its potential to zero (same as at infinity). How do your answers to (a) and (b) change?*

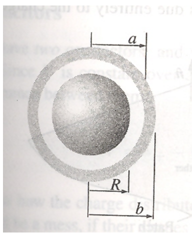
(a) $\boxed{\sigma_R = \frac{q}{4\pi R^2};\ \sigma_a = \frac{-q}{4\pi a^2};\ \sigma_b = \frac{q}{4\pi b^2}}$.

(b) $V(0) = -\int_\infty^0 \mathbf{E}\cdot d\mathbf{l} = -\int_\infty^b \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_b^a (0)dr - \int_a^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_R^0 (0)dr = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{q}{b} + \frac{q}{R} - \frac{q}{a}\right)}$.

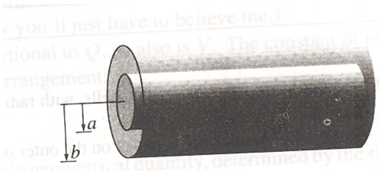
(c) $\boxed{\sigma_b \to 0}$ (the charge "drains off"); $V(0) = -\int_\infty^a (0)dr - \int_a^R \left(\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\right)dr - \int_R^0 (0)dr = \boxed{\frac{1}{4\pi\epsilon_0}\left(\frac{q}{R} - \frac{q}{a}\right)}$.



(a) Problem 3



(b) Problem 6



(c) Problem 7

7. **[G 2.39]** *Find the capacitance per unit length of two coaxial metal cylindrical tubes, of radii a and b (Fig.).*

Say the charge on the inner cylinder is Q, for a length L. The field is given by Gauss's law: $\int \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot L = \frac{1}{\epsilon_0}Q_{\text{enc}} = \frac{1}{\epsilon_0}Q \Rightarrow \mathbf{E} = \frac{Q}{2\pi\epsilon_0 L}\frac{1}{s}\hat{\mathbf{s}}$. Potential difference between the cylinders is

$$V(b) - V(a) = -\int_a^b \mathbf{E}\cdot d\mathbf{l} = -\frac{Q}{2\pi\epsilon_0 L}\int_a^b \frac{1}{s}ds = -\frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right).$$

As set up here, a is at the higher potential, so $V = V(a) - V(b) = \frac{Q}{2\pi\epsilon_0 L}\ln\left(\frac{b}{a}\right)$.

$C = \frac{Q}{V} = \frac{2\pi\epsilon_0 L}{\ln\left(\frac{b}{a}\right)}$, so capacitance *per unit length* is $\boxed{\frac{2\pi\epsilon_0}{\ln\left(\frac{b}{a}\right)}}$.

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8. **[G 2.42]** *If the electric field in some region is given (in spherical coordinates) by the expression* $\mathbf{E}\,(\mathbf{r})=\frac{A\hat{\mathbf{r}}+B\sin\theta\cos\phi\hat{\phi}}{r}$,
where A and B are constants, what is the charge density?

$$\rho = \epsilon_0\, \nabla\cdot\mathbf{E} = \epsilon_0 \left\{ \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{A}{r}\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(\frac{B\sin\theta\cos\phi}{r}\right)\right\}$$
$$= \epsilon_0\left[\frac{1}{r^2}A + \frac{1}{r\sin\theta}\frac{B\sin\theta}{r}(-\sin\phi)\right] = \boxed{\frac{\epsilon_0}{r^2}(A - B\sin\phi).}$$

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9. *A charge distribution on the surface of a sphere of radius R, is given by*

$$\sigma(\theta,\phi) = \sigma_0\sin\theta\sin\phi$$

Find the electric field and the potential on the z axis.

Let $\mathbf{r}=(0,0,z)$ and $\mathbf{r}'=(R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$. Then $\left|\mathbf{r}-\mathbf{r}'\right| = (R^2+z^2-2Rz\cos\theta)^{1/2}$.

$$\begin{aligned}
E_x &= -\frac{\sigma_0 R^2}{4\pi\epsilon_0}\int_0^\pi \frac{\sin^3\theta\, d\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}\underbrace{\int_0^{2\pi}\cos\phi\sin\phi\, d\phi}_{=0} = 0\\
E_y &= -\frac{\sigma_0 R^2}{4\pi\epsilon_0}\int_0^\pi \frac{\sin^3\theta\, d\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}\int_0^{2\pi}\sin^2\phi\, d\phi = \begin{cases} -\frac{\sigma_0 R^3}{3\epsilon_0 z^3} & z>R\\ -\frac{\sigma_0}{3\epsilon_0} & z<R\end{cases}\\
E_z &= \frac{\sigma_0 R^2}{4\pi\epsilon_0}\int_0^\pi \frac{(z-R\cos\theta)\sin^2\theta\, d\theta}{(R^2+z^2-2Rz\cos\theta)^{3/2}}\underbrace{\int_0^{2\pi}\sin\phi\, d\phi}_{=0} = 0
\end{aligned}$$

Now the θ integral in E_y is quiet messy, but has a nice compact expression.

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10. *Use Gauss's law to find electric field at any point due to the charge distribution*

   (a) $\rho = \frac{\rho_0 a}{r}$ *if* $r<a$ *and is zero otherwise;*
   (b) $\rho = \rho_0(1-r^2/a^2)$ *if* $r<a$ *and is zero otherwise.*

(a) Use a spherical Gaussian surface with centre at origin and radius r, then $\oint \mathbf{E}\cdot d\mathbf{S} = E\cdot 4\pi r^2$ and

$$q_{\text{enclosed}} = \rho_0 a\int_0^{\tilde{r}}\frac{1}{r}4\pi r^2 dr = 4\pi\rho_0 a\frac{\tilde{r}^2}{2}$$

where $\tilde{r}=r$ if $r<a$, else $\tilde{r}=a$ if $r>a$. Then

$$\mathbf{E} = \begin{cases}\frac{\rho_0 a}{2\epsilon_0}\hat{\mathbf{r}} & r<a\\ \frac{\rho_0 a^3}{2\epsilon_0 r^2}\hat{\mathbf{r}} & r>a\end{cases}$$

(b) Simillary,

$$q_{\text{enclosed}} = \rho_0\int_0^{\tilde{r}}\left(1-\frac{r^2}{a^2}\right)4\pi r^2 dr = 4\pi\rho_0\tilde{r}^3\left(\frac{1}{3}-\frac{1}{5}\frac{\tilde{r}^2}{a^2}\right)$$

Thus,

$$\mathbf{E} = \begin{cases}\frac{\rho_0 r}{3\epsilon_0}\left(1-\frac{3}{5}\frac{r^2}{a^2}\right)\hat{\mathbf{r}} & r<a\\ \frac{2\rho_0 a^3}{15\epsilon_0 r^2}\hat{\mathbf{r}} & r>a\end{cases}$$
