1. *[G 2.6] Find the electric field a distance z above the center of a flat circular disk of radius R (see figure), which carries a uniform surface charge σ. What does your formula give in the limit $R\to\infty$? Also check the case $z \gg R$.*

Break it into rings of radius r, and thickness dr, and use Prob. 2.5 to express the field of each ring. Total charge of a ring is $\sigma \cdot 2\pi r \cdot dr = \lambda \cdot 2\pi r$, so $\lambda = \sigma dr$ is the "line charge" of each ring.

$$E_{\text{ring}} = \frac{1}{4\pi\epsilon_0}\frac{(\sigma dr)2\pi r z}{(r^2+z^2)^{3/2}}; \quad E_{\text{disk}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma z \int_0^R \frac{r}{(r^2+z^2)^{3/2}}dr.$$

$$\boxed{\mathbf{E}_{\text{disk}} = \frac{1}{4\pi\epsilon_0}2\pi\sigma z\left[\frac{1}{z} - \frac{1}{\sqrt{R^2+z^2}}\right]\hat{\mathbf{z}}.}$$

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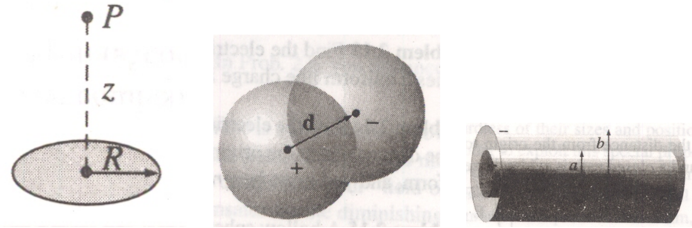
2. *A spherical surface of radius R and center at origin carries a surface charge $\sigma(\theta,\phi) = \sigma_0\cos\theta$. Find the electric field at z on z-axis. Treat the case $z < R$ (inside) as well as $z > R$ (outside). [Hint: Be sure to take the positive square root: $\sqrt{R^2+z^2-2Rz} = (R-z)$ if $R > z$, but its $(z-R)$ if $R < z$.]*

Let $\hat{\mathbf{r}} = z\hat{\mathbf{z}}$ and $\mathbf{r}' = R\left(\sin\theta'\cos\phi'\hat{\mathbf{x}} + \sin\theta'\sin\phi'\hat{\mathbf{y}} + \cos\theta'\hat{\mathbf{z}}\right)$.

$dS' = R^2\sin\theta'\,d\theta'\,d\phi'$

$\left|\mathbf{r}-\mathbf{r}'\right| = \left(R^2+z^2-2Rz\cos\theta'\right)^{1/2}$

$$\begin{aligned}
\mathbf{E}(z\hat{\mathbf{z}}) &= \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\mathbf{r}')(\mathbf{r}-\mathbf{r}')}{|\mathbf{r}-\mathbf{r}'|^3}dS \\
&= \frac{\sigma_0}{4\pi\epsilon_0}\int_S \frac{\cos\theta'(-R\sin\theta'\cos\phi'\hat{\mathbf{x}} - R\sin\theta'\sin\phi'\hat{\mathbf{y}} + (z-R\cos\theta')\hat{\mathbf{z}})}{(R^2+z^2-2Rz\cos\theta')^{3/2}}R^2\sin\theta'\,d\theta'\,d\phi' \\
&= \frac{\sigma_0 R^2}{2\epsilon_0}\hat{\mathbf{z}}\int_{-1}^{1}\frac{u(z-Ru)}{(R^2+z^2-2Rzu)^{3/2}}du \\
&= \frac{2\sigma_0 R^3}{3\epsilon_0 z^3}\hat{\mathbf{z}} \quad z > R
\end{aligned}$$



(a) Problem 1 (b) Problem 4 (c) Problem 5

Figure 1

$$= -\frac{\sigma_0}{3\epsilon_0}\hat{\mathbf{z}} \quad z < R$$

Note: **E** is constant inside sphere.

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3. *[G 2.9] Suppose the electric field in some region is found to be* $\mathbf{E} = kr^3\hat{\mathbf{r}}$, *in spherical coordinates (k is some constant).*

   (a) *Find the charge density* ρ.

   (b) *Find the total charge contained in the sphere of radius R, centered at the origin. (Do it two different ways.)*

(a) $\rho = \epsilon_0 \, \nabla \cdot \mathbf{E} = \epsilon_0 \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \cdot kr^3\right) = \epsilon_0 \frac{1}{r^2} k(5r^4) = \boxed{5\epsilon_0 kr^2.}$

(b) *By Gauss's law:* $Q_{\text{enc}} = \epsilon_0 \oint \mathbf{E} \cdot d\mathbf{a} = \epsilon_0(kR^3)(4\pi R^2) = \boxed{4\pi\epsilon_0 kR^5.}$

By direct integration: $Q_{\text{enc}} = \int \rho \, d\tau = \int_0^R (5\epsilon_0 kr^2)(4\pi r^2 dr) = 20\pi\epsilon_0 k \int_0^R r^4 dr = 4\pi\epsilon_0 kR^5$.

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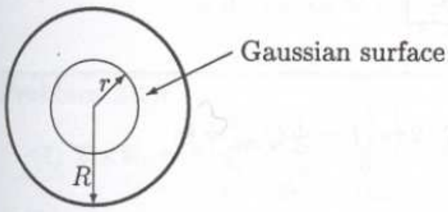
4. *[G 2.12] Use Gauss's law to find the electric field inside a uniformly charged sphere (charge density* ρ*). [G 2.18] Two spheres each of radius R and carrying uniform charge densities* $+\rho$ *and* $-\rho$*, respectively, are placed so that they partially overlap (See Figure). Call the vector from the positive center to the negative center* **d**. *Show that the field in the region of overlap is constant, and find its value.*

[G 2.12]



$\oint \mathbf{E} \cdot d\mathbf{a} = E \cdot 4\pi r^2 = \frac{1}{\epsilon_0} Q_{\text{enc}} = \frac{1}{\epsilon_0}\frac{4}{3}\pi r^3 \rho$. So

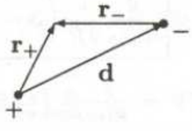
$$\boxed{\mathbf{E} = \frac{1}{3\epsilon_0}\rho r \hat{\mathbf{r}}.}$$

Since $Q_{\text{tot}} = \frac{4}{3}\pi R^2 \rho$, $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{Q}{R^3}\hat{\mathbf{r}}$

[G 2.18]

From Prob. 2.12, the field inside the positive sphere is $\mathbf{E}_+ = \frac{\rho}{3\epsilon_0}\mathbf{r}_+$, where $\mathbf{r}_+$ is the vector from the positive center to the point in question. Likewise, the field of the negative sphere is $-\frac{\rho}{3\epsilon_0}\mathbf{r}_-$. So the *total* field is

$$\mathbf{E} = \frac{\rho}{3\epsilon_0}(\mathbf{r}_+ - \mathbf{r}_-)$$



But (see diagram) $\mathbf{r}_+ - \mathbf{r}_- = \mathbf{d}$. So $\boxed{\mathbf{E} = \frac{\rho}{3\epsilon_0}\mathbf{d}.}$

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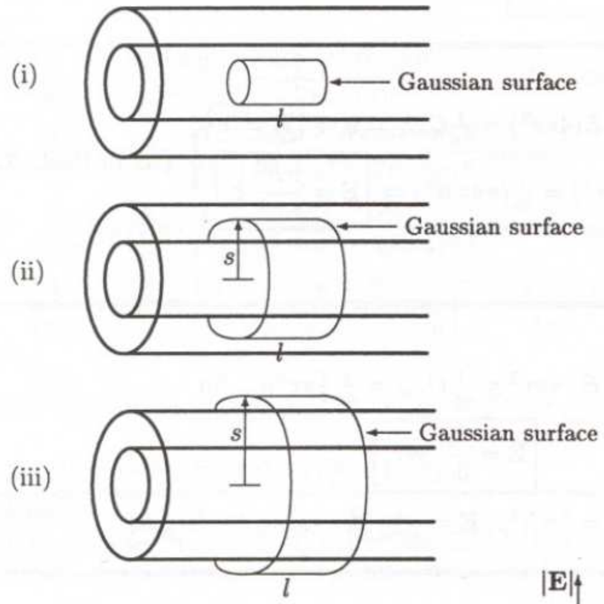
5. *[G 2.16] A long coaxial cable (see figure) carries a uniform* volume *charge density ρ on the inner cylinder (radius a), and a uniform surface charge density on the outer cylindrical shell (radius b). This surface charge is negative and of just the right magnitude so that the cable as a whole is electrically neutral. Find the electric field in each of the three regions: (a) inside the inner cylinder ($s < a$), (b) between the cylinders ($a < s < b$), (c) outside the cable ($s > b$). Plot $|\mathbf{E}|$ as a function of s.*



(i) $\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{enc} = \frac{1}{\epsilon_0}\rho\pi s^2 l;$

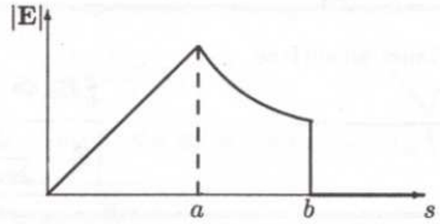
$$\boxed{\mathbf{E} = \frac{\rho s}{2\epsilon_0}\hat{\mathbf{s}}.}$$

(ii) $\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{enc} = \frac{1}{\epsilon_0}\rho\pi a^2 l;$

$$\boxed{\mathbf{E} = \frac{\rho a^2}{2\epsilon_0 s}\hat{\mathbf{s}}.}$$

(iii) $\oint \mathbf{E}\cdot d\mathbf{a} = E\cdot 2\pi s\cdot l = \frac{1}{\epsilon_0}Q_{enc} = 0;$

$$\boxed{\mathbf{E} = 0.}$$



6. *[G 2.20] One of these is an impossible electrostatic field. Which one?*

(a) $\mathbf{E} = k\,[xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3xz\hat{\mathbf{z}}]$

(b) $\mathbf{E} = k\,\left[y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}\right]$.

Here k is a constant with the appropriate units. For the possible one, find the potential, using the origin as your reference point. Check your answers by computing ∇V. [Hint: You must select a specific path to integrate along. It does not matter what path you choose, since the answer is path-independent, but you simply cannot integrate unless you have a particular path in mind.]

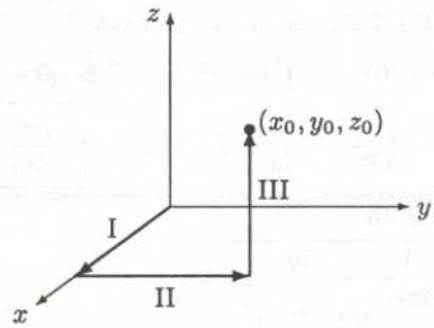
(1) $\nabla\times\mathbf{E}_1 = k\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ xy & 2yz & 3zx\end{vmatrix} = k\,[\hat{\mathbf{x}}(0-2y) + \hat{\mathbf{y}}(0-3z) + \hat{\mathbf{z}}(0-x)] \neq 0,$

so $\mathbf{E}_1$ is an *impossible* electrostatic field.

(2) $\nabla\times\mathbf{E}_2 = k\begin{vmatrix}\hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}}\\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z}\\ y^2 & 2xy+z^2 & 2yz\end{vmatrix} = k\,[\hat{\mathbf{x}}(2z-2z) + \hat{\mathbf{y}}(0-0) + \hat{\mathbf{z}}(2y-2y)] = 0,$

so $\mathbf{E}_2$ is a *possible* electrostatic field.

Let's go by the indicated path:



$\mathbf{E}\cdot d\mathbf{l} = (y^2\,dx + (2xy + z^2)dy + 2yz\,dz)k$

Step I: $y = z = 0$; $dy = dz = 0$. $\mathbf{E}\cdot d\mathbf{l} = ky^2\,dx = 0$.

Step II: $x = x_0$, $y: 0 \to y_0$, $z = 0$. $dx = dz = 0$.

$\mathbf{E}\cdot d\mathbf{l} = k(2xy + z^2)dy = 2kx_0y\,dy$.

$\int_{II}\mathbf{E}\cdot d\mathbf{l} = 2kx_0\int_0^{y_0} y\,dy = kx_0y_0^2$.

Step III: $x = x_0$, $y = y_0$, $z: 0 \to z_0$; $dx = dy = 0$.

$\mathbf{E}\cdot d\mathbf{l} = 2kyz\,dz = 2ky_0z\,dz$.

$\int_{III}\mathbf{E}\cdot d\mathbf{l} = 2y_0k\int_0^{z_0} z\,dz = ky_0z_0^2$.

$$V(x_0, y_0, z_0) = -\int_0^{(x_0,y_0,z_0)}\mathbf{E}\cdot d\mathbf{l} = -k(x_0y_0^2 + y_0z_0^2), \text{ or } \boxed{V(x,y,z) = -k(xy^2 + yz^2).}$$

Check: $-\nabla V = k\left[\frac{\partial}{\partial x}(xy^2+yz^2)\,\hat{\mathbf{x}} + \frac{\partial}{\partial y}(xy^2+yz^2)\,\hat{\mathbf{y}} + \frac{\partial}{\partial z}(xy^2+yz^2)\,\hat{\mathbf{z}}\right] = k[y^2\,\hat{\mathbf{x}} + (2xy+z^2)\,\hat{\mathbf{y}} + 2yz\,\hat{\mathbf{z}}] = \mathbf{E}$.

7. *[G 2.21] Find the potential inside and outside a uniformly charged solid sphere whose radius is R and whose total charge is q. Use infinity as your reference point. Compute the gradient of V in each region, and check that it yields the correct field. Sketch V(r).*

$$V(r) = -\int_{\infty}^{r} \mathbf{E}\cdot d\mathbf{l}. \quad \begin{cases} \text{Outside the sphere } (r > R): & \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\hat{\mathbf{r}}. \\ \text{Inside the sphere } (r < R): & \mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}. \end{cases}$$

So for $r > R$: $V(r) = -\int_{\infty}^{r}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{\bar{r}^2}\right)d\bar{r} = \frac{1}{4\pi\epsilon_0}q\left(\frac{1}{\bar{r}}\right)\Big|_{\infty}^{r} = \boxed{\frac{q}{4\pi\epsilon_0}\frac{1}{r}},$

and for $r < R$: $V(r) = -\int_{\infty}^{R}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{\bar{r}^2}\right)d\bar{r} - \int_{R}^{r}\left(\frac{1}{4\pi\epsilon_0}\frac{q}{R^3}\bar{r}\right)d\bar{r} = \frac{q}{4\pi\epsilon_0}\left[\frac{1}{R} - \frac{1}{R^3}\left(\frac{r^2-R^2}{2}\right)\right]$

$$= \boxed{\frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(3 - \frac{r^2}{R^2}\right).}$$

When $r > R$, $\nabla V = \frac{q}{4\pi\epsilon_0}\frac{\partial}{\partial r}\left(\frac{1}{r}\right)\hat{\mathbf{r}} = -\frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$, so $\mathbf{E} = -\nabla V = \frac{q}{4\pi\epsilon_0}\frac{1}{r^2}\hat{\mathbf{r}}$.

When $r < R$, $\nabla V = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\frac{\partial}{\partial r}\left(3 - \frac{r^2}{R^2}\right)\hat{\mathbf{r}} = \frac{q}{4\pi\epsilon_0}\frac{1}{2R}\left(-\frac{2r}{R^2}\right)\hat{\mathbf{r}} = -\frac{q}{4\pi\epsilon_0}\frac{r}{R^3}\hat{\mathbf{r}}$; so $\mathbf{E} = -\nabla V = \frac{1}{4\pi\epsilon_0}\frac{q}{R^3}r\hat{\mathbf{r}}$.

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8. *[G 2.22] Find the potential a distance s from an infinitely long straight wire that carries a uniform line charge λ. Compute the gradient of your potential, and check that it yields the correct field.*

$\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{2\lambda}{s}\hat{\mathbf{s}}$ (Prob. 2.13). In this case we cannot set the reference point at ∞, since the charge itself extends to ∞. Let's set it at $s = a$. Then

$V(s) = -\int_{a}^{s}\left(\frac{1}{4\pi\epsilon_0}\frac{2\lambda}{\bar{s}}\right)d\bar{s} = \boxed{-\frac{1}{4\pi\epsilon_0}2\lambda\ln\left(\frac{s}{a}\right).}$

(In this form it is clear why $a = \infty$ would be no good—likewise the other "natural" point, $a = 0$.)

$\nabla V = -\frac{1}{4\pi\epsilon_0}2\lambda\frac{\partial}{\partial s}\left(\ln\left(\frac{s}{a}\right)\right)\hat{\mathbf{s}} = -\frac{1}{4\pi\epsilon_0}2\lambda\frac{1}{s}\hat{\mathbf{s}} = -\mathbf{E}.$

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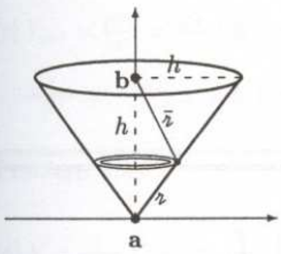
9. *[G 2.26] A conical surface (an empty ice-cream cone) carries a uniform surface charge σ. The height of the cone is h, as is the radius of the top. Find the potential difference between points* **a** *(the vertex) and* **b** *(the center of the top).*



$$V(\mathbf{a}) = \frac{1}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\left(\frac{\sigma 2\pi r}{\imath}\right)d\imath = \frac{2\pi\sigma}{4\pi\epsilon_0}\frac{1}{\sqrt{2}}(\sqrt{2}h) = \frac{\sigma h}{2\epsilon_0}.$$

(where $r = \imath/\sqrt{2}$)

$$V(\mathbf{b}) = \frac{1}{4\pi\epsilon_0}\int_0^{\sqrt{2}h}\left(\frac{\sigma 2\pi r}{\bar{\imath}}\right)d\imath, \quad \text{where } \bar{\imath} = \sqrt{h^2 + \imath^2 - \sqrt{2}h\imath}.$$

$$= \frac{2\pi\sigma}{4\pi\epsilon_0}\frac{1}{\sqrt{2}}\int_0^{\sqrt{2}h}\frac{\imath}{\sqrt{h^2+\imath^2-\sqrt{2}h\imath}}\,d\imath$$

$$= \frac{\sigma}{2\sqrt{2}\epsilon_0}\left[\sqrt{h^2+\imath^2-\sqrt{2}h\imath} + \frac{h}{\sqrt{2}}\ln(2\sqrt{h^2+\imath^2-\sqrt{2}h\imath} + 2\imath - \sqrt{2}h)\right]_0^{\sqrt{2}h}$$

$$= \frac{\sigma}{2\sqrt{2}\epsilon_0}\left[h + \frac{h}{\sqrt{2}}\ln(2h+2\sqrt{2}h-\sqrt{2}h) - h - \frac{h}{\sqrt{2}}\ln(2h-\sqrt{2}h)\right] = \frac{\sigma}{2\sqrt{2}\epsilon_0}\frac{h}{\sqrt{2}}\left[\ln(2h+\sqrt{2}h) - \ln(2h-\sqrt{2}h)\right]$$

$$= \frac{\sigma h}{4\epsilon_0}\ln\left(\frac{2+\sqrt{2}}{2-\sqrt{2}}\right) = \frac{\sigma h}{4\epsilon_0}\ln\left(\frac{(2+\sqrt{2})^2}{2}\right) = \frac{\sigma h}{2\epsilon_0}\ln(1+\sqrt{2}).$$

$$\therefore \boxed{V(\mathbf{a}) - V(\mathbf{b}) = \frac{\sigma h}{2\epsilon_0}\left[1 - \ln(1+\sqrt{2})\right].}$$

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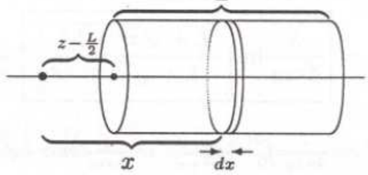
10. *[G2.27] Find the potential on the axis of a uniformly charged solid cylinder, a distance z from the center. The length of the cylinder is L and its radius is R, and the charge density is ρ. Use your result to calculate the electric field at this point. (Assume that $z > L/2$.)*



Cut the cylinder into slabs, as shown in the figure, and use result of Prob. 2.25c, with $z \to x$ and $\sigma \to \rho\, dx$:

$$V = \frac{\rho}{2\epsilon_0}\int_{z-L/2}^{z+L/2}\left(\sqrt{R^2+x^2}-x\right)dx$$

$$= \frac{\rho}{2\epsilon_0}\frac{1}{2}\left[x\sqrt{R^2+x^2}+R^2\ln(x+\sqrt{R^2+x^2})-x^2\right]\Big|_{z-L/2}^{z+L/2}$$

$$= \boxed{\frac{\rho}{4\epsilon_0}\left\{\left(z+\frac{L}{2}\right)\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}-\left(z-\frac{L}{2}\right)\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}+R^2\ln\left[\frac{z+\frac{L}{2}+\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}{z-\frac{L}{2}+\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}\right]-2zL\right\}}.$$

(*Note:* $-\left(z+\frac{L}{2}\right)^2+\left(z-\frac{L}{2}\right)^2 = -z^2-zL-\frac{L^2}{4}+z^2-zL+\frac{L^2}{4} = -2zL$.)

$$\mathbf{E} = -\nabla V = -\hat{\mathbf{z}}\frac{\partial V}{\partial z} = -\frac{\hat{\mathbf{z}}\rho}{4\epsilon_0}\left\{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}+\frac{\left(z+\frac{L}{2}\right)^2}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}-\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}-\frac{\left(z-\frac{L}{2}\right)^2}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}\right.$$

$$\left.+R^2\underbrace{\left[\frac{1+\frac{z+\frac{L}{2}}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}}{z+\frac{L}{2}+\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}-\frac{1+\frac{z-\frac{L}{2}}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}}{z-\frac{L}{2}+\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}\right]}_{\frac{1}{\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}}-\frac{1}{\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}}}-2L\right\}$$

$$\mathbf{E} = -\frac{\hat{\mathbf{z}}\rho}{4\epsilon_0}\left\{2\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}-2\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}-2L\right\}$$

$$= \boxed{\frac{\rho}{2\epsilon_0}\left[L-\sqrt{R^2+\left(z+\frac{L}{2}\right)^2}+\sqrt{R^2+\left(z-\frac{L}{2}\right)^2}\right]\hat{\mathbf{z}}.}$$