1. Curl of a gradient is

$$\begin{aligned}
\nabla \times (\nabla t) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial t}{\partial x} & \frac{\partial t}{\partial y} & \frac{\partial t}{\partial z} \end{vmatrix} \\
&= \hat{\mathbf{x}}\left(\frac{\partial^2 t}{\partial y \partial z} - \frac{\partial^2 t}{\partial z \partial y}\right) + \hat{\mathbf{y}}\left(\frac{\partial^2 t}{\partial z \partial x} - \frac{\partial^2 t}{\partial x \partial z}\right) + \hat{\mathbf{z}}\left(\frac{\partial^2 t}{\partial x \partial y} - \frac{\partial^2 t}{\partial y \partial x}\right) \\
&= 0 \quad \text{by equality of cross-derivatives} .
\end{aligned}$$

Now, $\nabla f = 2xy^3z^4\hat{\mathbf{x}} + 3x^2y^2z^4\hat{\mathbf{y}} + 4x^2y^3z^3\hat{\mathbf{z}}$, so

$$\begin{aligned}
\nabla \times (\nabla f) &= \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2xy^3z^4 & 3x^2y^2z^4 & 4x^2y^3z^3 \end{vmatrix} \\
&= \hat{\mathbf{x}}\left(3 \times 4x^2y^2z^3 - 4 \times 3x^2y^2z^3\right) + \hat{\mathbf{y}}\left(4 \times 2xy^3z^3 - 2 \times 4xy^3z^3\right) \\
&+ \hat{\mathbf{z}}\left(2 \times 3xy^2z^4 - 3 \times 2xy^2z^4\right) \\
&= 0.
\end{aligned}$$

2. The parametrization for given helix $C:\ \mathbf{r}(\phi) = \left(R\cos\phi, R\sin\phi, p\frac{\phi}{2\pi}\right)$.

$$|\mathbf{r}'(\phi)| = |(-R\sin\phi, R\cos\phi, p/2\pi)| = \frac{1}{2\pi}\left(4\pi R^2 + p^2\right)^{1/2}$$

The length of helix is

$$\int_C dl = \int_0^{2\pi} \mathbf{r}'(\phi) d\phi = \left(4\pi R^2 + p^2\right)^{1/2}.$$

3. The surface S and its projection R on the xy plane are shown in the figure below.
$\int\int_S \mathbf{A} \cdot \mathbf{n}\, dS = \int\int_R \mathbf{A} \cdot \mathbf{n}\, \frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|}$
To obtain $\mathbf{n}$ note that a vector perpendicular to the surface $2x + 3y + 6z = 12$ is given by $\nabla(2x + 3y + 6z) = 2\mathbf{i} + 3\mathbf{j} + 6\mathbf{k}$. Then a unit normal to any point of S is
$\mathbf{n} = \frac{2\mathbf{i}+3\mathbf{j}+6\mathbf{k}}{\sqrt{2^2+3^2+6^2}} = \frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}$
Thus $\mathbf{n} \cdot \mathbf{k} = \left(\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right)\cdot\mathbf{k} = \frac{6}{7}$ and so $\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \frac{7}{6}dxdy$.
Also $\mathbf{A} \cdot \mathbf{n} = (18z\mathbf{i} - 12\mathbf{j} + 3y\mathbf{k}) \cdot \left(\frac{2}{7}\mathbf{i} + \frac{3}{7}\mathbf{j} + \frac{6}{7}\mathbf{k}\right) = \frac{36z-36+18y}{7} = \frac{36-12x}{7}$,
using the fact that $z = \frac{12-2x-3y}{6}$ from the equation of S. Then
$\int\int_S \mathbf{A} \cdot \mathbf{n} dS = \int\int_R \mathbf{A} \cdot \mathbf{n}\frac{dxdy}{|\mathbf{n}\cdot\mathbf{k}|} = \int\int_R \left(\frac{36-12x}{7}\right) \frac{7}{6} dx = \int\int_R (6-2x) dxdy$
To evaluate this double integral over R, keep x fixed and integrate with respect to y from $y = 0$ to $y = \frac{12-2x}{3}$; the integrate with respect to x from $x = 0$ to $x = 6$. In this manner R is completely covered. The integral becomes
$\int_{x=0}^{6}\int_{y=0}^{(12-2x)/3}(6-2x)dydx = \int_{x=0}^{6}\left(24 - 12x + \frac{4x^2}{3}\right)dx = 24$
If we had chosen the positive unit normal $\mathbf{n}$ opposite to that in the figure, we would have obtained the result -24.

4. $\int T d\tau = \int z^2 dxdydz$. You can do the integrals in any order - here it is simplest to save z for last:

$$\int z^2 \left[\int \left(\int dx\right) dy\right] dz.$$

The sloping surface is $x + y + z = 1$, so the x integral is

$$\int_0^{(1-y-z)} dx = 1 - y - z$$

For a given z, y ranges from 0 to $1-z$, so the y integral is

$$\int_0^{(1-z)} (1-y-z)dy = \left[(1-z)y - (y^2/2)\right]\Big|_0^{(1-z)}$$
$$= (1-z)^2 - [(1-z)^2/2] = (1/2) - z + (z^2/2)$$

Finally, the z integral is

$$\int_0^1 z^2(\frac{1}{2} - z + \frac{z^2}{2})dz = \int_0^1 (\frac{z^2}{2} - z^3 + \frac{z^4}{2})dz = \boxed{1/60.}$$

5. $\nabla \times v = \hat{\mathbf{x}}(0-2y) + \hat{\mathbf{y}}(0-3z) + \hat{\mathbf{z}}(0-x) = -2y\hat{\mathbf{x}} - 3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}$.
$d\mathbf{a} = dydz\hat{\mathbf{x}}$, if we agree that the path integral shall run counterclockwise. So
$(\nabla \times \mathbf{v}) \cdot d\mathbf{a} = -2ydydz$.
$\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int \left\{ \int_0^{2-z} (-2y)dy \right\} dz$
$= -\int_0^2 (4 - 4z + z^2)dz = -(8 - 8 + \frac{8}{3}) = -\frac{8}{3}$
Meanwhile, $\mathbf{v} \cdot d\mathbf{l} = (xy)dx + (2yz)dy + (3zx)dz$. There are three segments.

   (a) $x = z = 0; dx = dz = 0. y: 0 \to 2. \int \mathbf{v} \cdot \mathbf{dl} = 0.$
   (b) $x = 0; z = 2 - y; dx = 0, dz = -dy, y: 2 \to 0. \mathbf{v} \cdot d\mathbf{l} = 2yzdy.$
   $\int \mathbf{v} \cdot \mathbf{dl} = \int_2^0 2y(2-y)dy = -\int_0^2 (4y - 2y^2)dy = -\frac{8}{3}.$
   (c) $x = y = 0; dx = dy = 0; z: 2 \to 0. \mathbf{v} \cdot \mathbf{dl} = 0. \int \mathbf{v} \cdot \mathbf{dl} = 0.$ So $\oint \mathbf{v} \cdot \mathbf{dl} = -\frac{8}{3}.$

6. $\nabla \cdot \mathbf{v} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 r\cos\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta r\sin\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(r\sin\theta\cos\phi)$
$= \frac{1}{r^2}3r^2\cos\theta + \frac{1}{r\sin\theta}r2\sin\theta\cos\theta + \frac{1}{r\sin\theta}r\sin\theta(-\sin\phi)$
$= 3\cos\theta + 2\cos\theta - \sin\phi = 5\cos\theta - \sin\phi$
$\int (\nabla \cdot \mathbf{v})d\tau = \int (5\cos\theta - \sin\phi)r^2\sin\theta drd\theta d\phi = \int_0^R r^2dr \int_0^{\frac{\pi}{2}} [\int_0^{2\pi}(5\cos\theta - \sin\phi)d\phi]d\theta\sin\theta$
$= (\frac{R^3}{3})(10\pi)\int_0^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta$
$= \frac{5\pi}{3}R^3$.
Two surfaces - one the hemisphere: $d\mathbf{a} = R^2\sin\theta d\theta d\phi\hat{\mathbf{r}}$; $r = R$; $\phi: 0 \to 2\pi$, $\theta: 0 \to \frac{\pi}{2}$.
$\int \mathbf{v} \cdot d\mathbf{a} = \int (r\cos\theta)R^2\sin\theta d\theta d\phi = R^3 \int_0^{\frac{\pi}{2}} \sin\theta\cos\theta d\theta \int_0^{2\pi} d\phi = R^3(\frac{1}{2})(2\pi) = \pi R^3$.
other the flat bottom: $d\mathbf{a} = (dr)(r\sin\theta d\phi)(+\hat{\theta}) = rdrd\phi\hat{\theta}$ (here $\theta = \frac{\pi}{2}$). r: $0 \to R$, $\phi: 0 \to 2\pi$.
$\int \mathbf{v} \cdot d\mathbf{a} = \int (r\sin\theta)(rdrd\phi) = \int_0^R r^2dr \int_0^{2\pi} d\phi = 2\pi\frac{R^3}{3}$.
Total: $\int \mathbf{v} \cdot d\mathbf{a} = \pi R^3 + \frac{2}{3}\pi R^3 = \frac{5}{3}\pi R^3$.

7. From the figure, $\hat{\mathbf{s}} = \cos\phi\hat{\mathbf{x}} + \sin\phi\hat{\mathbf{y}}$; $\hat{\phi} = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}$; $\hat{\mathbf{z}} = \hat{\mathbf{z}}$.
Multiply first by $\cos\phi$, second by $\sin\phi$, and substract:
$\hat{\mathbf{s}}\cos\phi - \hat{\phi}\sin\phi = \cos^2\phi\hat{\mathbf{x}} + \cos\phi\sin\phi\hat{\mathbf{y}} + \sin^2\phi\hat{\mathbf{x}} - \sin\phi\cos\phi\hat{\mathbf{y}} = \hat{\mathbf{x}}(\sin^2\phi + \cos^2\phi) = \hat{\mathbf{x}}$.
So $\hat{\mathbf{x}} = \cos\phi\hat{\mathbf{s}} - \sin\phi\hat{\phi}$.
Multiply first by $\sin\phi$, second by $\cos\phi$, and add:
$\hat{\mathbf{s}}\sin\phi + \hat{\phi}\cos\phi = \sin\phi\cos\phi\hat{\mathbf{x}} + \sin^2\phi\hat{\mathbf{y}} - \sin\phi\cos\phi\hat{\mathbf{x}} + \cos^2\phi\hat{\mathbf{y}} = \hat{\mathbf{y}}(\sin^2\phi + \cos^2\phi) = \hat{\mathbf{y}}$.
So $\hat{\mathbf{y}} = \sin\phi\hat{\mathbf{s}} + \cos\phi\hat{\phi}$. $\hat{\mathbf{z}} = \hat{\mathbf{z}}$.

8. Solutions are

   (a) $\int_{-2}^2 (2x+3)\frac{1}{3}\delta(x)dx = \frac{1}{3}(0+3) = 1.$
   (b) $\delta(1-x) = \delta(x-1)$, so $1 + 3 + 2 = 6$.
   (c) $\int_{-1}^1 9x^2\frac{1}{3}\delta(x+\frac{1}{3})dx = 9(-\frac{1}{3})^2\frac{1}{3} = \frac{1}{3}.$
   (d) 1 (if $a > b$), 0 (if $a < b$).