1. [**G1.8b**] A matrix $R_{3\times3}$ is said to be *orthogonal* (also called rotation) if

$$\sum_{i=1}^{3} R_{ji}R_{ki} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

This means, the rows of the matrix, when treated as vectors, are of unit length and are perpendicular to each other. Let (x, y, z) and (x', y', z') be the coordinates of a point in two different coordinate systems such that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Show that the coordinate transformation is orthogonal (that is, R is orthogonal) if and only if it preserves distance of every point from the origin (that is, $\sqrt{x'^2 + y'^2 + z'^2} = \sqrt{x^2 + y^2 + z^2}$).

2. [**G1.10c**] If $\mathbf{A}$ and $\mathbf{B}$ are two *vectors* (transform like coordinates under all orthogonal transformations) then

   (a) show that $\mathbf{A} \times \mathbf{B}$ is transforms like coordinates under coordinate transformation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

   (b) The *inversion* of coordinates is given by $x' = -x$, $y' = -y$ and $z' = -z$. Verify that inversion is orthogonal. How does $\mathbf{A} \times \mathbf{B}$ transform under inversion?

3. Consider a particle of mass m placed at (x, y, z). Moments of inertia of the particle about coordinate axes are defined as

$$I_x = m(y^2 + z^2) \quad I_y = m(x^2 + z^2) \quad \text{and} \quad I_z = m(x^2 + y^2).$$

Show that a quantity like (I_x, I_y, I_z) is not a vector by showing that it does not transform like coordinates under a rotaion about z axis by an angle ϕ.

4. [**G1.11**] Find the gradients of the following functions:

   (a) $f(x, y, z) = x^2 + y^3 + z^4$.
   (b) $f(x, y, z) = x^2y^2z^4$.
   (c) $f(x, y, z) = e^x \sin(y) \ln(z)$.
   (d) $f(x, y, z) = r^n$ where $r = \sqrt{x^2 + y^2 + z^2}$.

5. [**G1.12**] The height of a certain hill (in feet) is given by

$$h(x, y) = 10\left(2xy - 3x^3 - 4y^2 - 18x + 28y + 12\right)$$

where y is the distance (in miles) north, x the distance east of South Hadley.

   (a) Where is the top of the hill located?
   (b) How high is the hill?
   (c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In which direction is the slope steepest at that point?

6. **[G1.13]** Let (x_0, y_0, z_0) be a fixed point. Let $\mathbf{R} = \hat{\mathbf{x}}(x - x_0) + \hat{\mathbf{y}}(y - y_0) + \hat{\mathbf{z}}(z - z_0)$ and $R = |\mathbf{R}|$

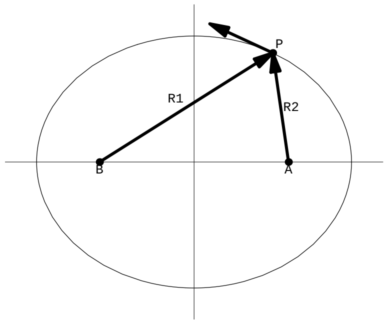

Figure 1: Problem 6

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   (a) Show that $\nabla R = \mathbf{R}/R$.

   (b) The figure shows an ellipse with foci at points A and B. Let P be a point on the ellipse. Show that lines AP and BP make equal angles with the tangent to the ellipse at P. [Hint: Use the fact that $R_1 + R_2 =$ Constant.]

7. **[G1.15]** Calculate the divergence of the following vector functions:

   (a) $\mathbf{v} = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$

   (b) $\mathbf{v} = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3xz\hat{\mathbf{z}}$

   (c) $\mathbf{v} = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$

8. **[G1.16]** Sketch the vector function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2}$$

   where $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ and compute its divergence. Explain.

9. **[G1.18]** Calculate curls of the vector functions in Prob 7.

10. **[G1.20]** Prove the following product rules:

$$\begin{aligned} \nabla (fg) &= f\nabla g + g\nabla f \\ \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \\ \nabla \times (f\mathbf{A}) &= f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f) \end{aligned}$$

Additional Problems [Not to be discussed in tutorials.]

1. Moment of intertia in fact is defined as

$$\begin{bmatrix} m(y^2+z^2) & -mxy & -mxz \\ -mxy & m(x^2+z^2) & -myz \\ -mxz & -myz & m(y^2+x^2) \end{bmatrix}.$$

   Show that moment of intertia transforms according to the equation 1.32 of Griffiths (3rd Ed) under a rotaion about z axis by an angle ϕ.

2. A matrix R is proper rotation if the determinant of R is equal to $+1$. Show that $\mathbf{A} \times \mathbf{B}$ transforms like coordinates under all proper rotations. [Hint: If R_1, R_2 and R_3 are rows of proper rotation R then $R_1 = R_2 \times R_3$.]

Solutions

1. It is enough to show that the square of distance is preserved. Now, let $x_1 = x$, $x_2 = y$ and $x_3 = z$ then

$$
\begin{aligned}
x'^2 + y'^2 + z'^2 &= \left(\sum_k R_{1k} x_k\right)^2 + \left(\sum_k R_{2k} x_k\right)^2 + \left(\sum_k R_{3k} x_k\right)^2 \\
&= \sum_i \left(\sum_k R_{ik} x_k\right)^2 \\
&= \sum_i \sum_k \sum_j R_{ik} R_{ij} x_k x_l \\
&= \sum_k \sum_l \left(\sum_i R_{ik} R_{il}\right) x_k x_l
\end{aligned}
$$

Now assume: R is orthogonal. Then $x'^2 + y'^2 + z'^2 = \sum_k x_k x_k = x^2 + y^2 + z^2$. To prove converse, assume that $x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2$ is true for all points. Thus it is true for $(1, 0, 0)$ then

$$1 = \sum_i R_{i1} R_{i1}$$

Simillarly, for $(1, 1, 0)$,

$$
\begin{aligned}
2 &= \left(\sum_i R_{i1} R_{i1}\right) + \left(\sum_i R_{i2} R_{i2}\right) + \left(\sum_i R_{i1} R_{i2}\right) + \left(\sum_i R_{i2} R_{i1}\right) \\
\implies 0 &= 2\left(\sum_i R_{i1} R_{i2}\right)
\end{aligned}
$$

2. [G1.10c]

   (a) Just consider x component of $\mathbf{A}' \times \mathbf{B}'$.

$$
\begin{aligned}
\left(\mathbf{A}' \times \mathbf{B}'\right)_x &= A'_y B'_z - A'_z B'_y \\
&= (-\sin\phi A_x + \cos\phi A_y) B_z - A_z(-\sin\phi B_x + \cos\phi B_y) \\
&= -\sin\phi \left(A_x B_z - A_z B_x\right) + \cos\phi \left(A_y B_z - A_z B_y\right) \\
&= \cos\phi \left(\mathbf{A} \times \mathbf{B}\right)_x + \sin\phi \left(\mathbf{A} \times \mathbf{B}\right)_y
\end{aligned}
$$

   (b) Verify orthogonality of inversion using relations of problem 1. Clearly $(\mathbf{A}' \times \mathbf{B}') = (\mathbf{A} \times \mathbf{B})$ .Thus cross product of two vectors is called a pseudovector.

3. Under rotation about z-axis,

$$
\begin{aligned}
I'_x &= m(z'^2 + y'^2) \\
&= m\left(z^2 + (x \sin\alpha + y \cos\alpha)^2\right) \\
\text{And} \quad I_x \cos\alpha - I_y \sin\alpha &\neq I'_x
\end{aligned}
$$

4. (c) $\nabla f = e^x \sin(y) \ln(z) \hat{\mathbf{x}} + e^x \cos(y) \ln(z) \hat{\mathbf{y}} + \frac{e^x \cos(y)}{z} \hat{\mathbf{y}}$;
   (d) $\nabla f = n r^{n-2} \mathbf{r}$

5. [Erratum: x^2 instead of x^3 in the expression of h) Gradient of h

$$\nabla h(x, y) = 10(2y - 6x - 18)\hat{\mathbf{x}} + 10\left(2x - 8y + 28\right)\hat{\mathbf{y}}$$

(a) Say, hill top is located at (x^*, y^*). Then $\nabla h(x^*, y^*) = 0$. This gives

$$\left.\begin{array}{r} 2y^* - 6x^* - 18 = 0 \\ 2x^* - 8y^* + 28 = 0 \end{array}\right\} \begin{array}{l} x^* = -2 \\ y^* = 3 \end{array}$$

The hilltop is at 3 miles north and 2 miles west of South Hadley.

(b) $h(-2, 3) = 720\text{ft}$.

(c) Putting in $x = 1, y = 1$, we get $\nabla h(1,1) = 220(-\hat{\mathbf{x}} + \hat{\mathbf{y}})$. Thus the slope is $200\sqrt{2}$ ft/mile and direction is northwest.

6. G1.13

(a) $R = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$, then

$$\frac{\partial R}{\partial x} = \frac{1}{2} \frac{2(x-x_0)}{\sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}} = \frac{\mathbf{R} \cdot \hat{\mathbf{x}}}{R}$$

Then $\nabla R = \mathbf{R}/R$. Notice that ∇R is a unit vector in the direction of $\mathbf{R}$.

(b) The ellipse is a level curve of function $f(x, y) = R_1 + R_2$, thus the gradient must be $\perp$ to the tangent $\mathbf{t}$ to the ellipse at P. So,$\mathbf{t} \cdot \nabla(R_1 + R_2) = 0 \implies \mathbf{t} \cdot \hat{\mathbf{R}}_1 = -\mathbf{t} \cdot \hat{\mathbf{R}}_2$ Hence lines AP and BP make equal angles with the tangent to the ellipse at P.

7. (a) $\nabla \cdot \left(x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}\right) = 0$
   (b) $\nabla \cdot (xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3xz\hat{\mathbf{z}}) = y + 2z + 3x$
   (c) $\nabla \cdot \left(y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}\right) = 2x + 2y$

8. When $\mathbf{r} \neq 0$, $\partial r/\partial x = x/r$

$$\begin{aligned} \frac{\partial}{\partial x}\left(\frac{x}{r^3}\right) &= \frac{1}{r^3} - \frac{3x}{r^4}\frac{\partial r}{\partial x} \\ &= \frac{1}{r^3} - \frac{3x^2}{r^5} \\ \text{Then} \quad \nabla \cdot \left(\frac{\mathbf{r}}{r^3}\right) &= \frac{3}{r^3} - \frac{3(x^2+y^2+z^2)}{r^5} = 0 \end{aligned}$$

Divergence of this vector field is 0 everywhere.

9. (a) $\nabla \times \mathbf{v} = \begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & 3xz^2 & -2xz \end{vmatrix} = -6xz\hat{\mathbf{x}} + 2z\hat{\mathbf{y}} + 3z^2\hat{\mathbf{z}}$
   (b) $\nabla \times \mathbf{v} = -2y\hat{\mathbf{x}} + -3z\hat{\mathbf{y}} - x\hat{\mathbf{z}}$
   (c) $\nabla \times \mathbf{v} = 0$

10. This is to be proven by brute force. No tricks.