

1. [G1.8b] A matrix $R_{3 \times 3}$ is said to be *orthogonal* (also called rotation) if

$$\sum_{i=1}^3 R_{ji}R_{ki} = \begin{cases} 1 & \text{if } j = k \\ 0 & \text{if } j \neq k. \end{cases}$$

This means, the rows of the matrix, when treated as vectors, are of unit length and are perpendicular to each other. Let (x, y, z) and (x', y', z') be the coordinates of a point in two different coordinate systems such that

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = R \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

Show that the coordinate transformation is orthogonal (that is, R is orthogonal) if and only if it preserves distance of every point from the origin (that is, $\sqrt{x'^2 + y'^2 + z'^2} = \sqrt{x^2 + y^2 + z^2}$).

2. [G1.10c] If \mathbf{A} and \mathbf{B} are two *vectors* (transform like coordinates under all orthogonal transformations) then

- (a) show that $\mathbf{A} \times \mathbf{B}$ transforms like coordinates under coordinate transformation

$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos \phi & \sin \phi & 0 \\ -\sin \phi & \cos \phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

- (b) The *inversion* of coordinates is given by $x' = -x$, $y' = -y$ and $z' = -z$. Verify that inversion is orthogonal. How does $\mathbf{A} \times \mathbf{B}$ transform under inversion?

3. Consider a particle of mass m placed at (x, y, z) . Moments of inertia of the particle about coordinate axes are defined as

$$I_x = m(y^2 + z^2) \quad I_y = m(x^2 + z^2) \quad \text{and} \quad I_z = m(x^2 + y^2).$$

Show that a quantity like (I_x, I_y, I_z) is not a vector by showing that it does not transform like coordinates under a rotation about z axis by an angle ϕ .

4. [G1.11] Find the gradients of the following functions:

- (a) $f(x, y, z) = x^2 + y^3 + z^4$.
 (b) $f(x, y, z) = x^2 y^2 z^4$.
 (c) $f(x, y, z) = e^x \sin(y) \ln(z)$.
 (d) $f(x, y, z) = r^n$ where $r = \sqrt{x^2 + y^2 + z^2}$.

5. [G1.12] The height of a certain hill (in feet) is given by

$$h(x, y) = 10(2xy - 3x^3 - 4y^2 - 18x + 28y + 12)$$

where y is the distance (in miles) north, x the distance east of South Hadley.

- (a) Where is the top of the hill located?
 (b) How high is the hill?
 (c) How steep is the slope (in feet per mile) at a point 1 mile north and one mile east of South Hadley? In which direction is the slope steepest at that point?

6. [G1.13] Let (x_0, y_0, z_0) be a fixed point. Let $\mathbf{R} = \hat{\mathbf{x}}(x - x_0) + \hat{\mathbf{y}}(y - y_0) + \hat{\mathbf{z}}(z - z_0)$ and $R = |\mathbf{R}|$

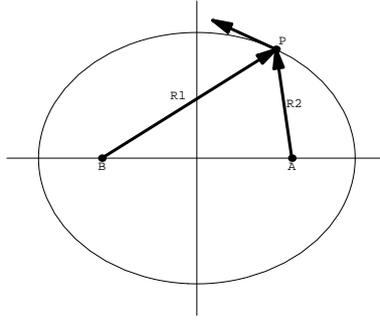


Figure 1: Problem 6

- (a) Show that $\nabla R = \mathbf{R}/R$.
- (b) The figure shows an ellipse with foci at points A and B . Let P be a point on the ellipse. Show that lines AP and BP make equal angles with the tangent to the ellipse at P . [Hint: Use the fact that $R_1 + R_2 = \text{Constant}$.]
7. [G1.15] Calculate the divergence of the following vector functions:
- (a) $\mathbf{v} = x^2\hat{\mathbf{x}} + 3xz^2\hat{\mathbf{y}} - 2xz\hat{\mathbf{z}}$
- (b) $\mathbf{v} = xy\hat{\mathbf{x}} + 2yz\hat{\mathbf{y}} + 3xz\hat{\mathbf{z}}$
- (c) $\mathbf{v} = y^2\hat{\mathbf{x}} + (2xy + z^2)\hat{\mathbf{y}} + 2yz\hat{\mathbf{z}}$
8. [G1.16] Sketch the vector function

$$\mathbf{v} = \frac{\hat{\mathbf{r}}}{r^2}$$

where $\mathbf{r} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$ and compute its divergence. Explain.

9. [G1.18] Calculate curls of the vector functions in Prob 7.
10. [G1.20] Prove the following product rules:

$$\begin{aligned}\nabla(fg) &= f\nabla g + g\nabla f \\ \nabla \cdot (\mathbf{A} \times \mathbf{B}) &= \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B}) \\ \nabla \times (f\mathbf{A}) &= f(\nabla \times \mathbf{A}) - \mathbf{A} \times (\nabla f)\end{aligned}$$

Additional Problems [Not to be discussed in tutorials.]

1. Moment of inertia in fact is defined as

$$\begin{bmatrix} m(y^2 + z^2) & -mxy & -mxz \\ -mxy & m(x^2 + z^2) & -myz \\ -mxz & -myz & m(y^2 + x^2) \end{bmatrix}.$$

Show that moment of inertia transforms according to the equation 1.32 of Griffiths (3rd Ed) under a rotation about z axis by an angle ϕ .

2. A matrix R is proper rotation if the determinant of R is equal to $+1$. Show that $\mathbf{A} \times \mathbf{B}$ transforms like coordinates under all proper rotations. [Hint: If R_1, R_2 and R_3 are rows of proper rotation R then $R_1 = R_2 \times R_3$.]