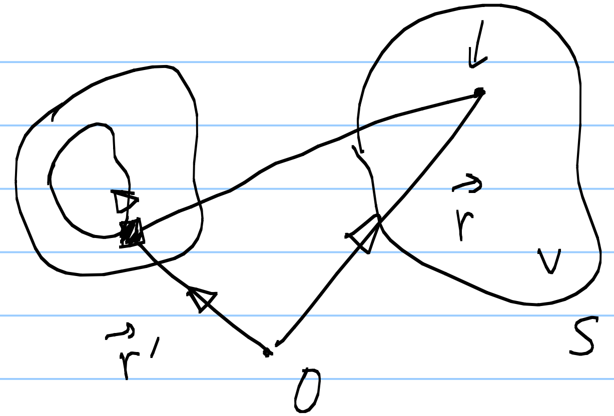
Magnetostatics (E)

Note Title 4/1/2009

(1) Divergence of B



$$B(\vec{r}) = \frac{\mu_0}{4\pi}\int dv'\, \frac{J(r') \times (r-r')}{|r-r'|^3}$$

$$\nabla \cdot B(\vec{r}) = \frac{\mu_0}{4\pi}\int dv'\; \nabla \cdot \left(\frac{J(r') \times (r-r')}{|r-r'|^3} \right)$$

$$\nabla \cdot (F \times G) = G \cdot (\nabla \times F) - F \cdot (\nabla \times G)$$

if F is ind of $\vec{r}$

$$= - F \cdot (\nabla \times G)$$

$$\left.\begin{aligned} F &= J(\vec{r}\,') \\ G &= \frac{(r-r')}{|r-r'|^3} \end{aligned}\right\}$$

$$\nabla \cdot \left(J(r') \times \frac{(r-r')}{|r-r'|^3} \right) = - J(r') \cdot \underbrace{\left(\nabla \times \frac{(r-r')}{|r-r'|^3} \right)}_{0}$$

$$= 0$$

$$\boxed{\nabla \cdot B(\vec{r}) = 0}$$

$$\int_V \nabla \cdot B \, dv = 0$$

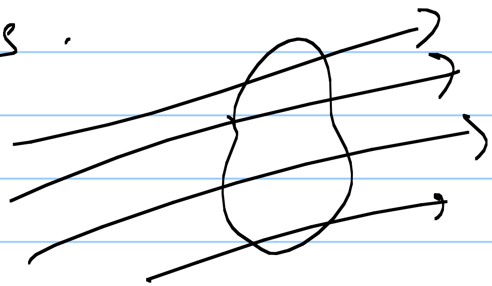
$$\boxed{\oint_S \bar{B} \cdot d\bar{s} = 0}$$

Compare with Gauss Law

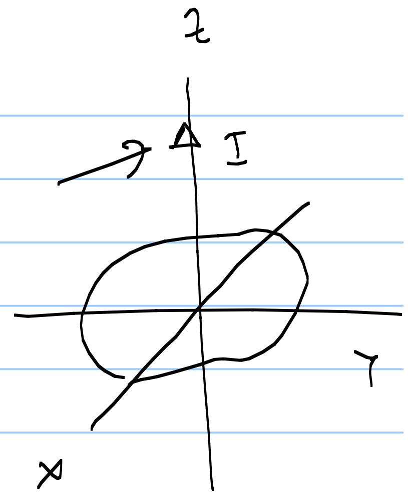
$$\oint E \cdot ds = \frac{1}{\epsilon_0} Q_{enc}$$

→ Magnetic monopoles don't Exist !

→ Magnetic field lines endless.



(ii) Curl of $\vec{B}$

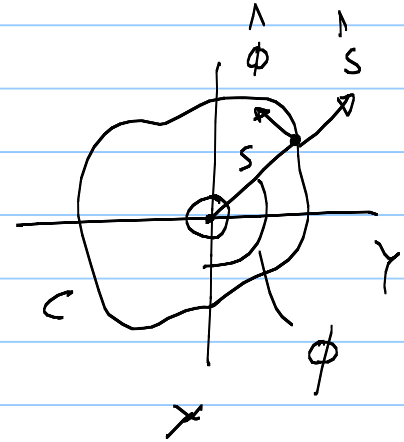


Example: St line carrying current I

$$\vec{B}(s) = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

Curve C : $\vec{r}(\phi) = (s(\phi), \phi, 0)$ cyl co-ordinates

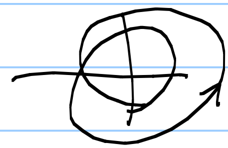
$\phi : 0 \to 2\pi$



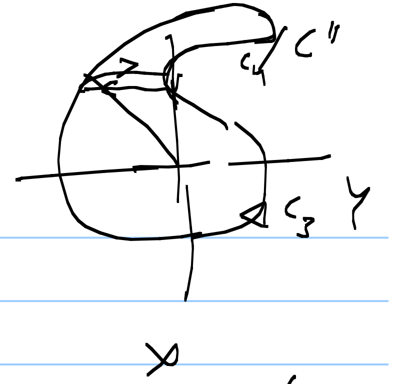
$$\vec{r}(\phi) = s(\phi)\,\hat{s}(\phi)$$

$$d\vec{r} = \left(\frac{ds}{d\phi}\hat{s} + s\frac{d\hat{s}}{d\phi}\right)d\phi + \frac{dz}{d\phi}\hat{z}\,d\phi$$

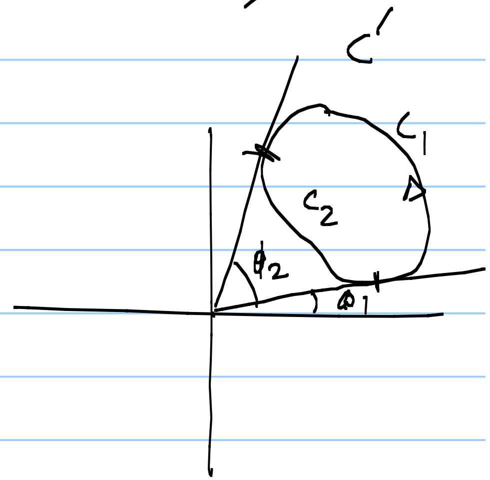
$$= \left(\frac{ds}{d\phi}\hat{s} + s\hat{\phi}\right)d\phi$$



$$\oint_C \vec{B}\cdot d\vec{r} = \underbrace{\int_0^{2\pi} \left(\frac{\mu_0 I}{2\pi s}\right)\hat{\phi}\cdot\overbrace{\left[\frac{ds}{d\phi}\hat{s} + s\hat{\phi}\right]} d\phi}$$



$$= \frac{\mu_0 I}{2\pi}\int_0^{2\pi} d\phi = \mu_0 I$$



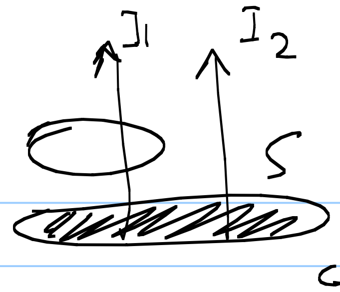
$C_1 : \phi : \phi_1 \to \phi_2$

$C_2 : \phi : \phi_2 \to \phi_2$

$$\oint_C \bar{B}\cdot d\bar{r} = \int_{C_1} \vec{B}\cdot d\bar{r} + \int_{C_2} \bar{B}\cdot d\bar{r}$$

$$= \frac{\mu_0 I}{2\pi}\left[\int_{\phi_1}^{\phi_2} d\phi + \int_{\phi_2}^{\phi_1} d\phi\right] = 0$$

$$\oint_C B\cdot d\vec{l} = \mu_0 I_{enc}$$



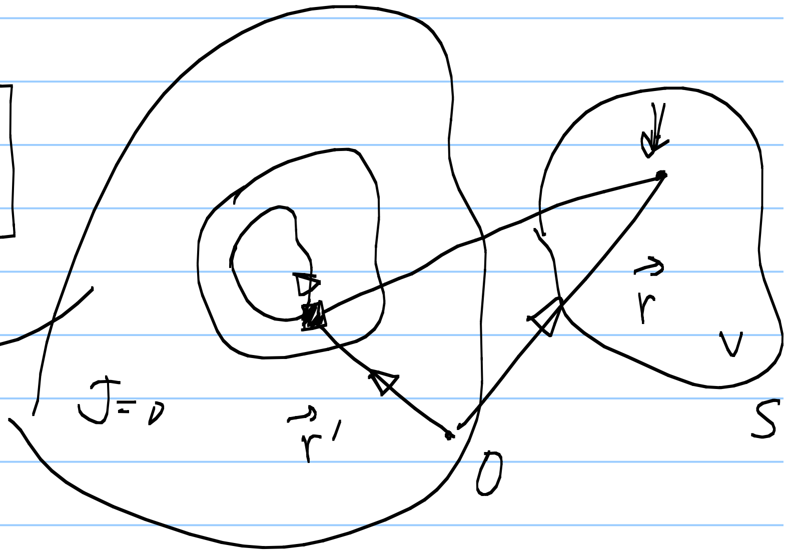
$$\int_S (\nabla\times B)\cdot ds = \mu_0 \int_S \vec{J}\cdot d\vec{s}$$

$$\boxed{\nabla\times B = \mu_0 J}$$

St. line current

(a)

$$\vec{\nabla}\times\vec{B} = \frac{\mu_0}{4\pi}\int_V dv' \;\nabla\times\left[\underbrace{J(r')}_{F}\times\underbrace{\frac{(r-r')}{|r-r'|^3}}_{G}\right]$$



$$\nabla \times (F \times G) = (G \cdot \nabla) F - (F \cdot \nabla) G + F(\nabla \cdot G) - G(\nabla \cdot F)$$

$$= F(\nabla \cdot G) - (F \cdot \nabla) G \quad \text{if } F \text{ is ind of } \vec{r}$$

$$\nabla \times \left[J(r') \times \frac{(r-r')}{|r-r'|^3} \right] = J(\vec{r}') \underbrace{\left[\nabla \cdot \frac{(r-r')}{|r-r'|^3} \right]}_{4\pi \delta^3 (r-r')} - \left(J(r') \cdot \nabla \right) \frac{r-r'}{|r-r'|^3}$$

$$\nabla \times B = \underline{\frac{\mu_0}{4\pi} \int J(r')\, 4\pi \delta^3 (r-r')\, dv'} - \underbrace{\frac{\mu_0}{4\pi} \int \left(\underline{J(\vec{r}') \cdot \nabla} \right) \left(\frac{r-r'}{|r-r'|^3} \right) dv'}$$

x component

HW

$$\underline{\nabla \times B(\vec{r}) = \mu_0 J(\vec{r})} + \underline{\oint_S \underline{J(r')} \frac{x-x'}{|r-r'|^3} \cdot ds}$$

bounded in space

0 if At ∞ $J \to 0$

$$\boxed{\begin{aligned}\nabla \times B &= \mu_0 J \\ \oint_c B \cdot d\vec{r} &= \mu_0 I_{enc}\end{aligned}}$$

Ampere's Law

Magnetostatics

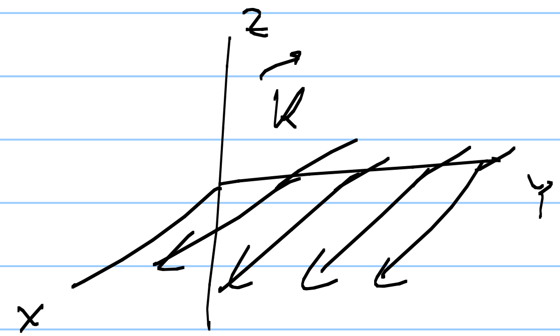
Given J

$$\boxed{\begin{aligned}\nabla \cdot B &= 0 \\ \nabla \times B &= \mu_0 J\end{aligned}}$$

Example

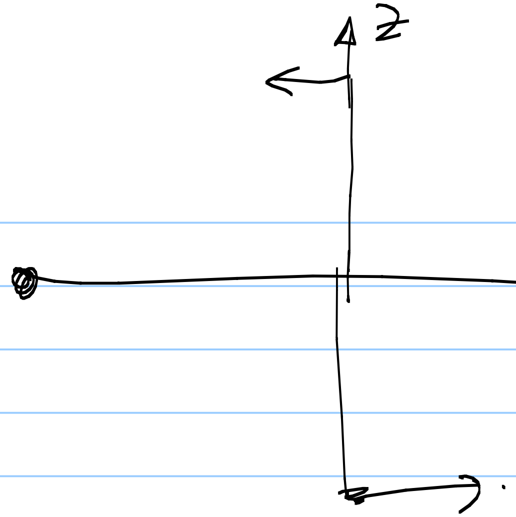
Sheet of current

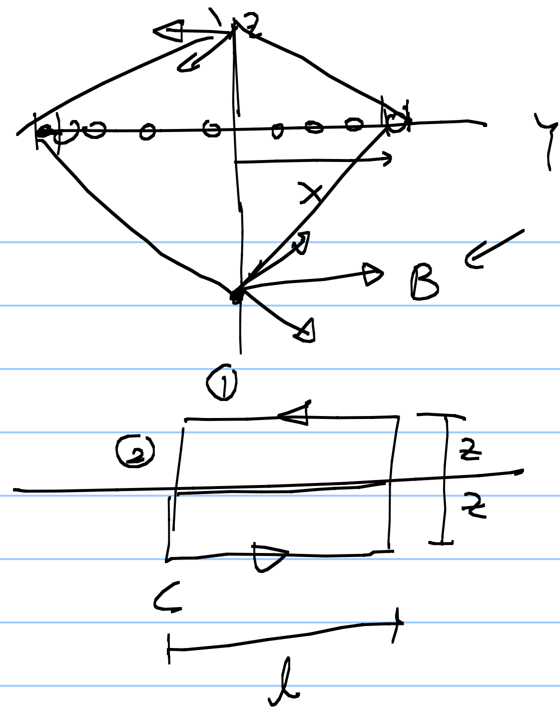
Surface current density $\vec{K} = K\,\hat{x}$



$\vec{B}$ (z only)

Fix the direction of B





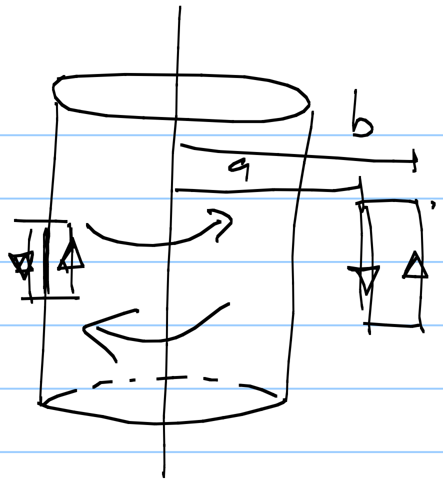
$$\int_C B \cdot d\vec{r} = B(z) \cdot l + 0 + B(z) \cdot l + 0$$

$$= \mu_0 K l$$

$$\Rightarrow \vec{B} = \frac{\mu_0 k}{2} (-\hat{y}) \qquad z > 0$$

$$= \frac{\mu_0 k}{2} (\hat{y}) \qquad z < 0$$

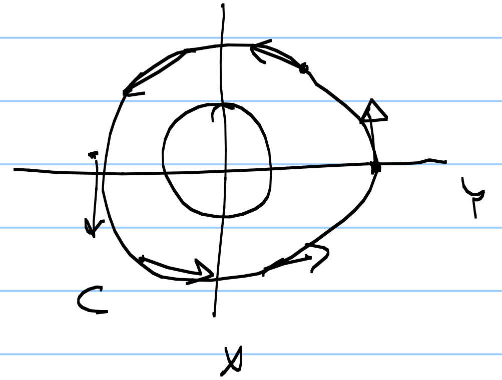
Ex Solenoid $\quad n \ \frac{\text{turns}}{\text{length}}, \ I$



$\vec{B}(s)$

$\oint_C \vec{B} \cdot d\vec{r} > 0$

$\Rightarrow \vec{B}$ cannot in $\hat{\phi}$



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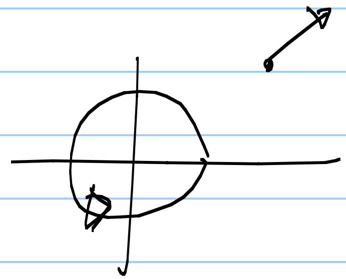
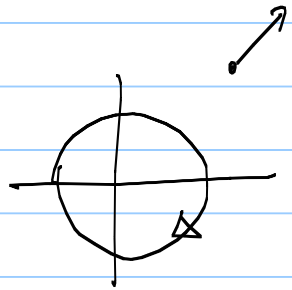




B is not in $\hat{s}$ direction

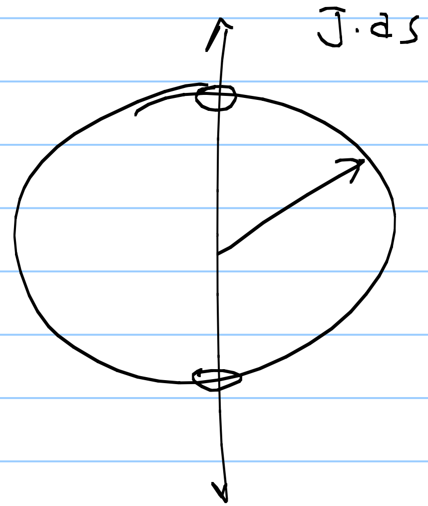
$$\vec{B} = B(s)\,\hat{z}$$

$$-B(a)\cdot l + B(b)\cdot l = 0$$

$$\Rightarrow \quad B(a) = B(b) = B(\infty) = 0 \quad \text{outside}$$

$$\vec{B} = \mu_0 n I\,\hat{z} \quad \text{inside}$$

$$\int \frac{J\cdot ds\,(x-x')}{r^3}$$



Magnetic Vector Potential

In ES: $\nabla \times E = 0 \Rightarrow E = -\nabla V$ $\qquad \nabla \times (\nabla \phi) = 0$

In MS $\nabla \cdot B = 0 \Rightarrow B = \nabla \times \bar{A}$ $\qquad \nabla \cdot (\nabla \times \bar{A}) = 0$

Magnetic Vector Potential.

$\vec{A}$ — Utility of computing $\vec{B}$

— theoretical value

Is $\vec{A}$ unique?

given E $\quad V_1(\vec{r})$

$$\underline{V_2(\vec{r}) = V_1(\vec{r}) + C}$$

~~For~~ given B $\quad A_1(\vec{r})$

$$\underline{A_2(\vec{r}) = A_1(\vec{r}) + \nabla \lambda(\vec{r})}$$

λ : any scalar field.

$$\nabla \times A_2 = \nabla \times A_1 + \underbrace{(\nabla \times \nabla \lambda)}_{0} = B$$

$\Rightarrow$ Choosing $\nabla \cdot A = 0$ $\leftarrow$

Suppose A_1 is such $\nabla \cdot A_1 \neq 0$

$$A_2 = A_1 + \nabla \lambda$$

$$\nabla \cdot A_2 = \nabla \cdot A_1 + \nabla^2 \lambda = 0$$

$$\nabla^2 \lambda = - \nabla \cdot A_1$$ Poisson Eq.

Differential Eq for Vector Potential

$$\nabla \times B = \mu_0 J$$

$$\Rightarrow \quad \nabla \times (\nabla \times A) = \mu_0 J$$

$$\Rightarrow \quad \nabla (\underline{\nabla \cdot A}) - \nabla^2 A = \mu_0 J$$

$$\Rightarrow \quad \nabla^2 \vec{A} = -\mu_0 \vec{J} \leftarrow$$

Three Poisson Eqns

$$\nabla^2 A_x = -\mu_0 J_x$$
$$\nabla^2 A_y = -\mu_0 J_y$$
$$\nabla^2 A_z = -\mu_0 J_z$$

only in Cartesian System

$$\nabla^2 A_r = -\mu_0 J_r$$

$$\nabla^2 (A_{r} \hat{r} + \cdots)$$

$(\vec{r})$ (θ, ϕ)

<u>If J is bounded.</u>

$$A_x = \frac{\mu_0}{4\pi} \int \frac{J_x(r')\, dv'}{|r - r'|}$$

In

$$\nabla^2 V = -\rho/\epsilon_0$$

J

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(r')\, dv'}{|r - r'|}$$

Strictly in Cartesian system

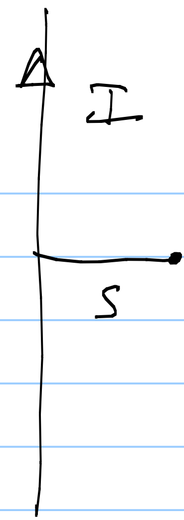
$$\vec{A} = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(r')\, dv'}{|r - r'|}$$

→ $\vec{A}$ is Parallel to $\vec{J}$?

→ $\vec{A}$ does follows J in typical situation

Ex St. long wire, I.

$$\bar{B}(s) = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$



$$\vec{A}(s) = A_z(s)\,\hat{z}$$

$$\bar{\nabla}\times\bar{A} = -\hat{\phi}\frac{\partial A_z}{\partial s} = \frac{\mu_0 I}{2\pi s}\hat{\phi}$$

in Cylindrical.

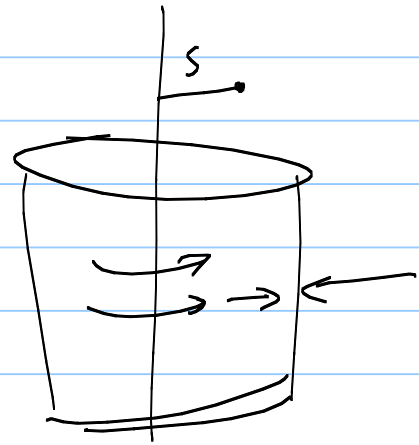
$$A_z = -\frac{\mu_0 I}{2\pi}\ln s$$

$$\vec{A} = -\frac{\mu_0 I}{2\pi}\ln s\,\hat{z}$$

$$\vec{\nabla}\cdot\vec{A} = \frac{\partial A_z}{\partial z} = 0$$

Ex Solenoid Current $\hat{\phi}$

$$\vec{A} = A_\phi(s)\,\hat{\phi}$$



$$\bar{\nabla} \times \bar{A} = \frac{1}{s}\frac{\partial}{\partial s}(sA_\phi)\,\hat{z} = \mu_0 n I\,\hat{z} \quad \text{(inside)} \qquad \text{in cyl}$$

$$A_\phi = \frac{\mu_0 n I s}{2} \qquad\qquad sA_\phi = \frac{\mu_0 n I s^2}{2}$$

$$\vec{A} = \frac{\mu_0 n I s}{2}\,\hat{\phi} \qquad \text{(inside)}$$

$$\vec{\nabla} \cdot A = 0 \quad \text{(check)}$$

$\vec{A}$ outside

$$\frac{1}{s}\frac{\partial}{\partial s}(sA_\phi) = 0$$

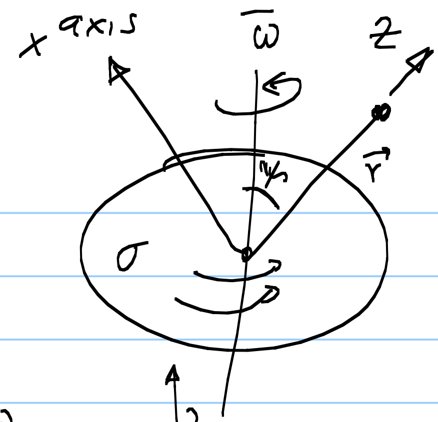
$$A_\phi = \frac{c}{s}$$

$$A_\phi = \frac{\mu_0 n I a^2}{2s}$$

Ex Spherical Shell: Radius R

Surface charge density σ

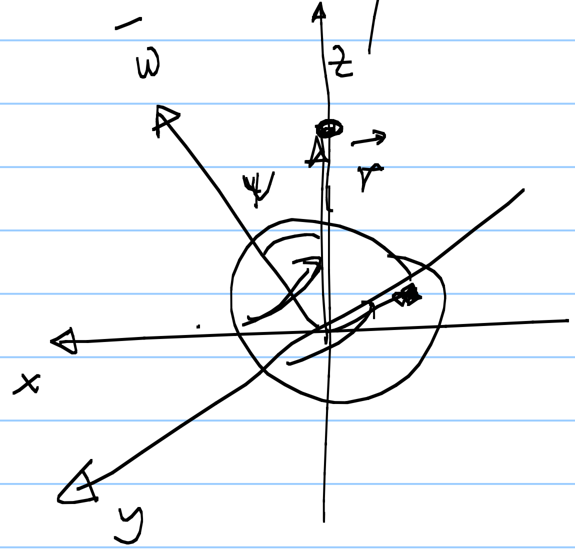
Spinning with ang velocity $\vec{\omega}$



$$\vec{r} = r\,\hat{z}$$

$$\vec{\omega} = (\omega\sin\psi,\ 0,\ \omega\cos\psi)$$

$$\vec{r}\,' = \left(R\sin\theta'\cos\phi',\ R\sin\theta'\sin\phi',\ R\cos\theta'\right)$$



sph. (R, θ', ϕ')

$$da' = R^2\sin\theta'\,d\theta'\,d\phi'$$

$$\vec{K} = \sigma\,\vec{v} = \sigma(\vec{\omega}\times\vec{r}\,') = \text{Surface current density}$$

$$= \sigma R\omega\Big[\hat{x}(-\cos\psi\sin\theta'\sin\phi') + \hat{y}(-\sin\psi\cos\theta' + \sin\theta'\cos\phi'\cos\psi) + \hat{z}(\sin\psi\sin\theta'\sin\phi')\Big]$$

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{k}(\vec{r}\,')\, da'}{|r - r'|} \qquad \int_0^{2\pi} \sin\phi'\, d\phi'$$

$$|r - r'| = (R^2 + r^2 - 2rR\cos\theta')^{1/2}$$

$$= \frac{-\mu_0}{4\pi} \sigma R^3 \omega\, \hat{y} \int \frac{\sin\psi \cos\theta' \sin\theta'\, d\theta'\, d\phi'}{(R^2 + r^2 - 2rR\cos\theta')^{1/2}}$$

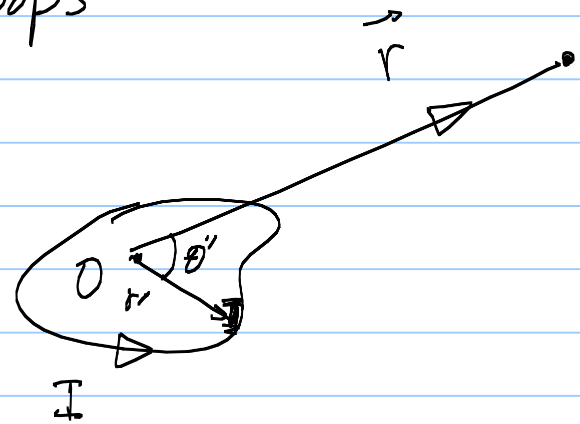
$$= -\frac{\mu_0 \sigma R^3 \omega\, \hat{y}}{2} \sin\psi \int_{-1}^{1} \frac{u\, du}{(R^2 + u^2 - 2rR)^{1/2}}$$

Vector Potential

Note Title 4/4/2009

Multipole Expansion $1/r$ only loops

$$\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \oint \frac{I \, \vec{dl}'}{|\vec{r}-\vec{r}'|} \quad \rightarrow K da'$$



$$\frac{1}{|\vec{r}-\vec{r}'|} = \frac{1}{r} \sum_{n=0}^{\infty} \left(\frac{r'}{r}\right)^n P_n(\cos\theta')$$

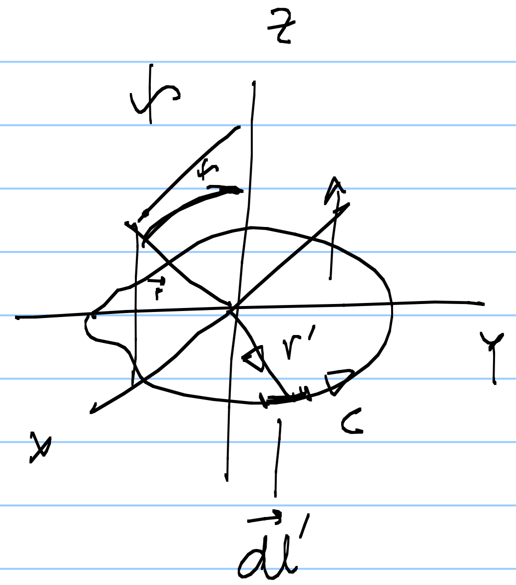
$$= \frac{1}{r} + \frac{r'}{r^2}\cos\theta' + \frac{r'^2}{r^3}\left(\frac{3\cos^2\theta' - 1}{2}\right) + \cdots$$

$$\vec{A}(\vec{r}) = \underbrace{\frac{\mu_0 I}{4\pi r} \oint \vec{dl}}_{\text{Monopole term}} + \underbrace{\frac{\mu_0 I}{4\pi r^2} \oint r'\cos\theta' \, \vec{dl}'}_{\text{dipole term}} + \underbrace{\cdots}_{\text{Q'pole term}}$$

Monopole term = 0 $\Rightarrow$ Monopole Moment = 0

Monopole don't Exist !

$$A_{dip}(\vec{r}) = \frac{\mu_0 I}{4\pi r^2} \oint r' \cos\theta' \, \vec{dl}' \Rightarrow$$



Planar loop: C in ~~X~~Y plane $(r', \theta = \frac{\pi}{2}, \phi')$

Angle between $\vec{r}$ and $\vec{r}'$

$$\cos\theta' = \sin\psi \cos\phi'$$

$$\vec{dl}' = dx' \hat{x} + dy' \hat{y}$$

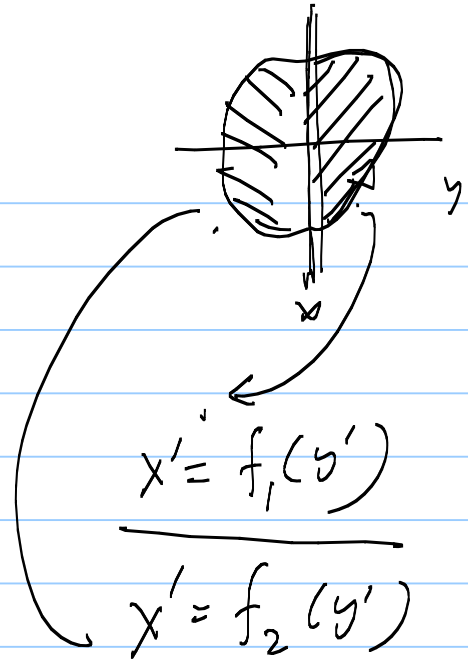
$$\vec{A}_{dip} = \frac{\mu_0 I \sin\psi}{4\pi r^2} \oint r' \cos\phi' (dx' \hat{x} + dy' \hat{y})$$

$$\vec{r} = (r\sin\psi, 0, r\cos\psi)$$

$$\vec{r}' = (r'\cos\phi', r'\sin\phi', 0)$$

$$= \frac{\mu_0 I \sin\psi}{4\pi r^2}\left[\hat{x}\oint x'\,dx' + \hat{y}\oint x'\,dy'\right]$$

$$= \frac{\mu_0 I \sin\psi}{4\pi r^2}\Big[0 + \hat{y}\underbrace{\oint x'\,dy'}_{\text{total area of loop} = a}\Big]$$

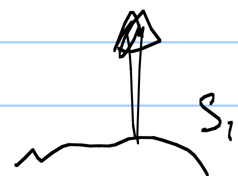


$$= \frac{\mu_0 \; I a \; \sin\psi \; \hat{y}}{4\pi r^2}$$

$\vec{m} = I(a\hat{z})$ = dipolemoment

$$\underbrace{= \frac{\mu_0}{4\pi}\,\frac{\vec{m}\times\hat{r}}{r^2}}_{\text{Co-ordinate free form}}$$

$\vec{r} = \hat{x}\, r\sin\psi + \hat{z}\, r\cos\psi$



Non planar loops (tut problem)

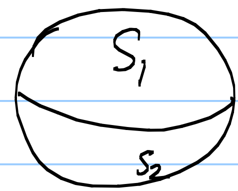
$$A_{dip}(\vec{r}) = \frac{\mu_0}{4\pi} \frac{\vec{m} \times \hat{r}}{r^2}$$

$$\vec{m} = I \int_S d\vec{a}$$ = magnetic dipole mom

$S_1 + S_2$ is closed

$$\int_{S_1+S_2} d\vec{a} = 0$$

$$\int_{S_1} d\vec{a} = - \int_{S_2} d\vec{a}$$

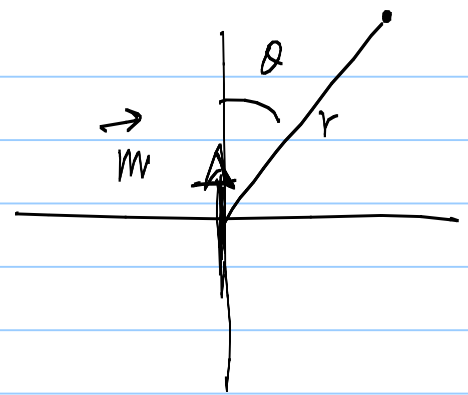


$$\oint da = 4\pi R^2$$

$$\oint d\vec{a} = 0$$

Pure dipole is where only dipole term is nonzero

$\vec{m} = m\hat{z}$

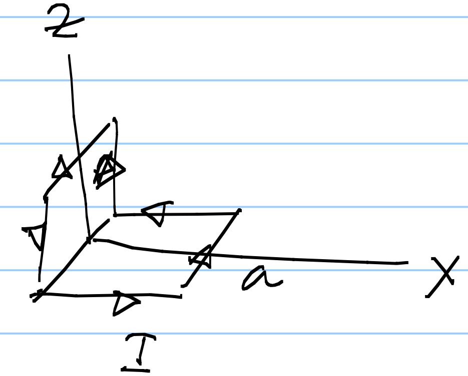


$$A(r,\theta,\phi) = \frac{\mu_0}{4\pi}\frac{m\sin\theta}{r^2}\hat{\phi}$$

$$B(r,\theta,\phi) = \nabla \times A$$

$$= \frac{\mu_0}{4\pi}\frac{m}{r^3}\left[\hat{r}\,2\cos\theta + \hat{\theta}\sin\theta\right]$$

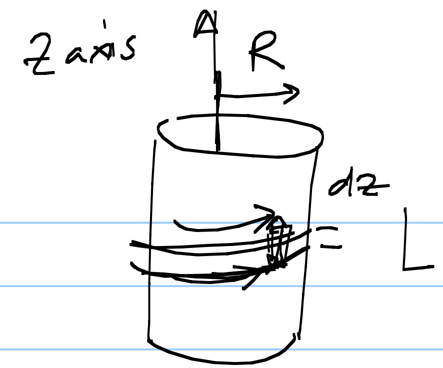
$\vec{m}$ is additive



<u>Ex</u> $\vec{m} = Ia^2\left[\hat{z} + \hat{x}\right]$

Ex Solenoid of L and

nI



$$\vec{m} = nI\pi R^2 L\,\hat{z}$$

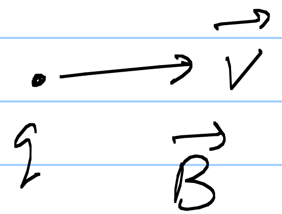
$\int da'$

$\left[\int d\bar{l}' \right] dz$

Lorentz Force Law

Expt Fact (Law)

A point charge in magnetic field

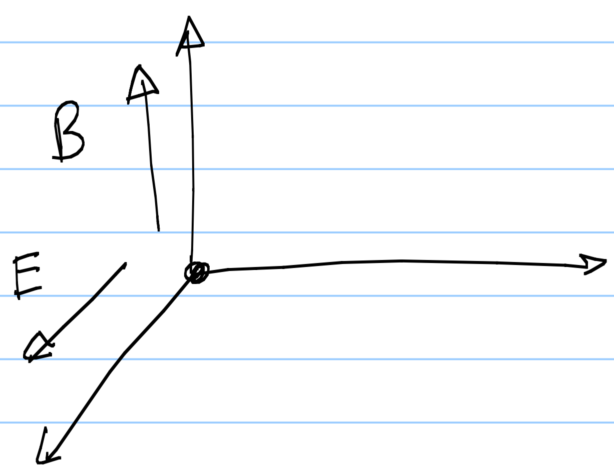


$$F_{mag} = q(\vec{v} \times \vec{B})$$

if E present

$$F_L = q(E + \bar{v} \times \vec{B})$$

Ex



Given $\vec{B} = B\hat{z}$ $\quad E = E\hat{x}$

Point Charge q, mass m is released from origin.
Find the trajectory:

point changes

$$\vec{B} = \frac{\mu_0}{4\pi} \frac{q v \times (r-r')}{|r-r'|}$$

if a point charge q has velocity $\vec{v}$

Field due to pt. Charge

?

$$\vec{r} = (x(t), y(t), z(t)) \qquad \vec{v} = (v_x(t), v_y(t), v_z(t))$$

$$m\vec{a} = q(E + \vec{v} \times \vec{B}) \qquad \vec{v} \times \vec{B} = \begin{bmatrix} \hat{x} & \hat{y} & \hat{z} \\ v_x & v_y & v_z \\ 0 & 0 & B \end{bmatrix}$$

$$= \hat{x}(v_y B) - v_x B \hat{y}$$

$$m(\ddot{x}\hat{x} + \ddot{y}\hat{y} + \ddot{z}\hat{z}) = \hat{x}(qE + qV_y B) + (-V_x B q)\hat{y}$$

$$\ddot{z} = 0 \Rightarrow z = At + B \qquad z(0) = 0, \quad \dot{z}(0) = 0$$

$$z = 0 \quad \forall t$$

$$\dot{v}_y = -\frac{qB}{m} V_x$$

$$\dot{V}_x = \frac{qE}{m} + \frac{qB}{m} V_y$$

$$\ddot{V}_x = \left(\frac{qB}{m}\right) \dot{V}_y = -\left(\frac{qB}{m}\right)^2 V_x = -\omega^2 V_x$$

$$V_x = A \sin(\omega t) + B \cos(\omega t) \qquad V_x(0) = 0$$

$$= A \sin(\omega t)$$

$$V_y = \frac{A\omega \cos \omega t}{\omega} - \frac{qE}{m\omega}$$

$$= A \cos \omega t - \frac{E}{B} \qquad V_y(0) = 0$$

$$= \frac{E}{B}(\cos \omega t - 1)$$

Cycloid.

$$\dot{x} = \frac{E}{B} \sin \omega t \qquad x(t) \rightarrow$$