Physics II

Electromagnetism and Optics

Charudatt Kadolkar

Indian Institute of Technology Guwahati

Jan 2009

Charge Distributions

There are no point charges or smooth charge distributions.

Charge Distributions

There are no point charges or smooth charge distributions.

Example

Consider two situations:

- A: Two "Point" charges, of magnitude q each, are located at $(0, d, 0)$ and $(0, -d, 0)$.
- B: One "Point" charge of magnitude $2q$ at $(0, 0, 0)$

Charge Distributions

There are no point charges or smooth charge distributions.

Example

Consider two situations:

A: Two "Point" charges, of magnitude q each, are located at $(0, d, 0)$ and $(0, -d, 0)$.

B: One "Point" charge of magnitude $2q$ at $(0, 0, 0)$

The electric field at a point $(x, 0, 0)$ is given by

$$\mathbf{E}_A = \frac{q}{4\pi\epsilon_0}\left[\frac{(x\hat{\mathbf{x}} - d\hat{\mathbf{y}})}{(d^2 + x^2)^{3/2}} + \frac{(x\hat{\mathbf{x}} + d\hat{\mathbf{y}})}{(d^2 + x^2)^{3/2}}\right]$$

Charge Distributions

There are no point charges or smooth charge distributions.

Example

Consider two situations:

- A: Two "Point" charges, of magnitude q each, are located at $(0, d, 0)$ and $(0, -d, 0)$.
- B: One "Point" charge of magnitude $2q$ at $(0, 0, 0)$

The electric field at a point $(x, 0, 0)$ is given by

$$\begin{aligned} \mathbf{E}_A &= \frac{q}{4\pi\epsilon_0}\left[\frac{(x\hat{\mathbf{x}} - d\hat{\mathbf{y}})}{(d^2 + x^2)^{3/2}} + \frac{(x\hat{\mathbf{x}} + d\hat{\mathbf{y}})}{(d^2 + x^2)^{3/2}}\right] \\ &= \frac{1}{4\pi\epsilon_0}\frac{(2q)\, x\hat{\mathbf{x}}}{(d^2 + x^2)^{3/2}} \end{aligned}$$

Charge Distributions

There are no point charges or smooth charge distributions.

Example

Consider two situations:

- A: Two "Point" charges, of magnitude q each, are located at $(0, d, 0)$ and $(0, -d, 0)$.
- B: One "Point" charge of magnitude $2q$ at $(0, 0, 0)$

The electric field at a point $(x, 0, 0)$ is given by

$$\begin{aligned}
\mathbf{E}_A &= \frac{q}{4\pi\epsilon_0}\left[\frac{(x\hat{\mathbf{x}} - d\hat{\mathbf{y}})}{(d^2 + x^2)^{3/2}} + \frac{(x\hat{\mathbf{x}} + d\hat{\mathbf{y}})}{(d^2 + x^2)^{3/2}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{(2q)\, x\hat{\mathbf{x}}}{(d^2 + x^2)^{3/2}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{(2q)\,\hat{\mathbf{x}}}{x^2}\left[1 - \frac{3}{2}\frac{d^2}{x^2} + \cdots\right]
\end{aligned}$$

Charge Distributions

There are no point charges or smooth charge distributions.

Example

Consider two situations:

- A: Two "Point" charges, of magnitude q each, are located at $(0, d, 0)$ and $(0, -d, 0)$.
- B: One "Point" charge of magnitude $2q$ at $(0, 0, 0)$

The electric field at a point $(x, 0, 0)$ is given by

$$\begin{aligned}
\mathbf{E}_A &= \frac{q}{4\pi\epsilon_0}\left[\frac{(x\hat{\mathbf{x}} - d\hat{\mathbf{y}})}{(d^2 + x^2)^{3/2}} + \frac{(x\hat{\mathbf{x}} + d\hat{\mathbf{y}})}{(d^2 + x^2)^{3/2}}\right] \\
&= \frac{1}{4\pi\epsilon_0}\frac{(2q)\,x\hat{\mathbf{x}}}{(d^2 + x^2)^{3/2}} \\
&= \frac{1}{4\pi\epsilon_0}\frac{(2q)\,\hat{\mathbf{x}}}{x^2}\left[1 - \frac{3}{2}\frac{d^2}{x^2} + \cdots\right]
\end{aligned}$$

And

$$\mathbf{E}_B = \frac{1}{4\pi\epsilon_0}\frac{(2q)\,\hat{\mathbf{x}}}{x^2}$$

Charge Distributions

Example

If $d = 1\,\text{mm}$ and $x = 1\,\text{m}$ then

$$\left|\frac{\mathbf{E}_A - \mathbf{E}_B}{\mathbf{E}_B}\right| \approx 10^{-6}$$

Charge Distributions

Example

If $d = 1\,\text{mm}$ and $x = 1\,\text{m}$ then

$$\left| \frac{\mathbf{E}_A - \mathbf{E}_B}{\mathbf{E}_B} \right| \approx 10^{-6}$$

- As far as measurements are concerned, modelling two distinct charges as a single point charge is fine.

Charge Distributions

Example

If $d = 1\,\text{mm}$ and $x = 1\,\text{m}$ then

$$\left|\frac{\mathbf{E}_A - \mathbf{E}_B}{\mathbf{E}_B}\right| \approx 10^{-6}$$

- As far as measurements are concerned, modelling two distinct charges as a single point charge is fine.
- Fundamental Laws? But Electromagnetism is not about charges and currents but about electric and magnetic field. We have "correct" laws for fields!

Charge Distributions

- Volume charge density is defined as

$$\rho(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V}.$$

Charge Distributions

- Volume charge density is defined as

$$\rho(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V}.$$

- Can $\rho(\mathbf{r}) = 0$ in any material? It is common to model $\rho(\mathbf{r}) = 0$ inside conductors! In reality, ΔV must be much larger than the atomic sizes. Say, of the order of one cubic μm.

Charge Distributions

- Volume charge density is defined as

$$\rho(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V}.$$

- Can $\rho(\mathbf{r}) = 0$ in any material? It is common to model $\rho(\mathbf{r}) = 0$ inside conductors! In reality, ΔV must be much larger than the atomic sizes. Say, of the order of one cubic μm.
- This will work if we are measuring fields outside the materials.

Charge Distributions

- Volume charge density is defined as

$$\rho(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V}.$$

- Can $\rho(\mathbf{r}) = 0$ in any material? It is common to model $\rho(\mathbf{r}) = 0$ inside conductors! In reality, ΔV must be much larger than the atomic sizes. Say, of the order of one cubic μm.
- This will work if we are measuring fields outside the materials.
- Surprisingly, such averaging works inside materials, too!

Charge Distributions

Four types of distributions

- 3D charge distributions: volume charge density

$$\rho(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V}.$$

Charge Distributions

Four types of distributions

- 3D charge distributions: volume charge density

$$\rho(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V}.$$

- 2D charge distributions: surface charge density, $\sigma(\mathbf{r})$.

Charge Distributions

Four types of distributions

- 3D charge distributions: volume charge density

$$\rho(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V}.$$

- 2D charge distributions: surface charge density, $\sigma(\mathbf{r})$.
- 1D charge distributions: linear charge density, $\lambda(\mathbf{r})$.

Charge Distributions

Four types of distributions

- 3D charge distributions: volume charge density

$$\rho(\mathbf{r}) = \lim_{\Delta V \to 0} \frac{\Delta Q}{\Delta V}.$$

- 2D charge distributions: surface charge density, $\sigma(\mathbf{r})$.
- 1D charge distributions: linear charge density, $\lambda(\mathbf{r})$.
- 0D charge distributions: point charges.

Charge Distributions

Example

Let $\rho(\mathbf{r}) = \rho_0$.

- Are there any point charges in this distribution?

Charge Distributions

Example

Let $\rho(\mathbf{r}) = \rho_0$.

- Are there any point charges in this distribution?
- How much charge at a point?

Charge Distributions

Example

Let $\rho(\mathbf{r}) = \rho_0$.

- Are there any point charges in this distribution?
- How much charge at a point?
- How much charge on the surface of a sphere of radius R?

Charge Distributions

Example

Let $\rho(\mathbf{r}) = \rho_0$.

- Are there any point charges in this distribution?
- How much charge at a point?
- How much charge on the surface of a sphere of radius R?

All charge distributions can, eventually, be represented as volume charge densities.

Charge Distributions

Example

Let $\rho(\mathbf{r}) = \rho_0$.

- Are there any point charges in this distribution?
- How much charge at a point?
- How much charge on the surface of a sphere of radius R?

All charge distributions can, eventually, be represented as volume charge densities.

Example

Volume charge density of

- a point charge q at $\mathbf{r}_0 = (x_0, y_0, z_0)$

$$\begin{aligned} \rho(\mathbf{r}) &= q\delta^3(\mathbf{r} - \mathbf{r}_0) \\ &= q\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \end{aligned}$$

Charge Distributions

Example

Let $\rho(\mathbf{r}) = \rho_0$.

- Are there any point charges in this distribution?
- How much charge at a point?
- How much charge on the surface of a sphere of radius R?

All charge distributions can, eventually, be represented as volume charge densities.

Example

Volume charge density of

- a point charge q at $\mathbf{r}_0 = (x_0, y_0, z_0)$

$$\begin{aligned} \rho(\mathbf{r}) &= q\delta^3(\mathbf{r} - \mathbf{r}_0) \\ &= q\delta(x - x_0)\delta(y - y_0)\delta(z - z_0) \end{aligned}$$

- a uniform surface charge density σ_0 on xy-plane

$$\rho(\mathbf{r}) = \sigma_0\delta(z)$$

Example

Let S be a spherical surface given by $r = R$. Surface charge density $\sigma(\theta, \phi) = \sigma_0 \cos\theta$. Find the total charge on upper hemisphere.

If S' is upper hemisphere, total charge is given by

$$Q = \int_{S'} \sigma(\mathbf{r})\, ds$$

Charge Distributions

Example

Let S be a spherical surface given by $r = R$. Surface charge density $\sigma(\theta, \phi) = \sigma_0 \cos\theta$. Find the total charge on upper hemisphere.

If S' is upper hemisphere, total charge is given by

$$Q = \int_{S'} \sigma(\mathbf{r})\, ds$$

Remember: Elementary area of spherical surface in spherical coordinates is $ds = R^2 \sin\theta\, d\theta\, d\phi$. Then

$$Q = \int_{S'} \sigma_0 \cos\theta\, R^2 \sin\theta\, d\theta\, d\phi$$

Example

Let S be a spherical surface given by $r = R$. Surface charge density $\sigma(\theta, \phi) = \sigma_0 \cos\theta$. Find the total charge on upper hemisphere.

If S' is upper hemisphere, total charge is given by

$$Q = \int_{S'} \sigma(\mathbf{r})\, ds$$

Remember: Elementary area of spherical surface in spherical coordinates is $ds = R^2 \sin\theta\, d\theta\, d\phi$. Then

$$\begin{aligned} Q &= \int_{S'} \sigma_0 \cos\theta\, R^2 \sin\theta\, d\theta\, d\phi \\ &= \frac{\sigma_0 R^2}{2} \int_0^{\pi/2} \sin 2\theta\, d\theta \int_0^{2\pi} d\phi \end{aligned}$$

Example

Let S be a spherical surface given by $r = R$. Surface charge density $\sigma(\theta, \phi) = \sigma_0 \cos\theta$. Find the total charge on upper hemisphere.

If S' is upper hemisphere, total charge is given by

$$Q = \int_{S'} \sigma(\mathbf{r})\, ds$$

Remember: Elementary area of spherical surface in spherical coordinates is $ds = R^2 \sin\theta\, d\theta\, d\phi$. Then

$$\begin{aligned} Q &= \int_{S'} \sigma_0 \cos\theta\, R^2 \sin\theta\, d\theta\, d\phi \\ &= \frac{\sigma_0 R^2}{2} \int_0^{\pi/2} \sin 2\theta\, d\theta \int_0^{2\pi} d\phi \\ &= \frac{\sigma_0 R^2}{2} \left[-\frac{\cos 2\theta}{2} \right]_0^{\pi/2} (2\pi) = \pi \sigma_0 R^2. \end{aligned}$$

Total charge on S is zero.

Electrostatics

- Interaction (forces) between two point particles depend, not only on their charges and positions, but also on their velocities and accelerations.
- However when particles are at rest, a relatively simple form for interaction emerges.
- Such simple form is also applicable when charges are moving at **very low** speeds and accelerations.
- It is relevant to study **Electrostatics** from application point of view.

Coulomb's Law

Coulomb's Law

Let q_1 and q_2 be two point charges located at $\mathbf{r}_1$ and $\mathbf{r}_2$ resp. Then the force exerted by q_1 on q_2 is

$$\mathbf{F}_{21} = k\, q_1 q_2 \, \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}.$$

Coulomb's Law

Coulomb's Law

Let q_1 and q_2 be two point charges located at $\mathbf{r}_1$ and $\mathbf{r}_2$ resp. Then the force exerted by q_1 on q_2 is

$$\mathbf{F}_{21} = k\, q_1 q_2 \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}.$$

- In SI units, $k = 1/4\pi\epsilon_0$, where ϵ_0 is called **permittivity of free space**. Its value is exactly known

$$\epsilon_0 = \frac{1}{\mu_0 c^2} \approx 8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N\ m}}.$$

Coulomb's Law

Let q_1 and q_2 be two point charges located at $\mathbf{r}_1$ and $\mathbf{r}_2$ resp. Then the force exerted by q_1 on q_2 is

$$\mathbf{F}_{21} = k\, q_1 q_2 \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}.$$

- In SI units, $k = 1/4\pi\epsilon_0$, where ϵ_0 is called **permittivity of free space**. Its value is exactly known

$$\epsilon_0 = \frac{1}{\mu_0 c^2} \approx 8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N\ m}}.$$

- Based on experiments, CL is believed to hold from 10^{-18} m (indirect evidence) to 10^7 m (geomagnetic measurements)

Coulomb's Law

Let q_1 and q_2 be two point charges located at $\mathbf{r}_1$ and $\mathbf{r}_2$ resp. Then the force exerted by q_1 on q_2 is

$$\mathbf{F}_{21} = k\, q_1 q_2 \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}.$$

- In SI units, $k = 1/4\pi\epsilon_0$, where ϵ_0 is called **permittivity of free space**. Its value is exactly known

$$\epsilon_0 = \frac{1}{\mu_0 c^2} \approx 8.85 \times 10^{-12} \frac{\mathrm{C}^2}{\mathrm{N\ m}}.$$

- Based on experiments, CL is believed to hold from 10^{-18} m (indirect evidence) to 10^{7} m (geomagnetic measurements)
- If $\mathbf{F} \propto 1/r^{2+\epsilon}$, experimental evidence puts limit $|\epsilon| < (2.7 \pm 3.1) \times 10^{-16}$ (laboratory).

Coulomb's Law

Let q_1 and q_2 be two point charges located at $\mathbf{r}_1$ and $\mathbf{r}_2$ resp. Then the force exerted by q_1 on q_2 is

$$\mathbf{F}_{21} = k\, q_1 q_2 \,\frac{(\mathbf{r}_2 - \mathbf{r}_1)}{|\mathbf{r}_2 - \mathbf{r}_1|^3}.$$

- In SI units, $k = 1/4\pi\epsilon_0$, where ϵ_0 is called **permittivity of free space**. Its value is exactly known

$$\epsilon_0 = \frac{1}{\mu_0 c^2} \approx 8.85 \times 10^{-12}\,\frac{\mathrm{C}^2}{\mathrm{N\ m}}.$$

- Based on experiments, CL is believed to hold from 10^{-18} m (indirect evidence) to 10^{7} m (geomagnetic measurements)
- If $\mathbf{F} \propto 1/r^{2+\epsilon}$, experimental evidence puts limit $|\epsilon| < (2.7 \pm 3.1) \times 10^{-16}$ (laboratory).
- Also linked to mass of photon, believed that $m_\gamma < 4 \times 10^{-51}$ kg (geomagnetic).

Linear Superposition

Force $\mathbf{F}_{AB}$ on a charge, say A, due to another charge, say B, is independent of presence of a third charge, say C. Total force on A is given by

$$\mathbf{F} = \mathbf{F}_{AB} + \mathbf{F}_{AC}.$$

- Easily generalize to several charges.

Linear Superposition

Force $\mathbf{F}_{AB}$ on a charge, say A, due to another charge, say B, is independent of presence of a third charge, say C. Total force on A is given by

$$\mathbf{F} = \mathbf{F}_{AB} + \mathbf{F}_{AC}.$$

- Easily generalize to several charges.
- Experimental evidence verified this law to accuracy of 10^{-4}.

Linear Superposition

Linear Superposition

Force $\mathbf{F}_{AB}$ on a charge, say A, due to another charge, say B, is independent of presence of a third charge, say C. Total force on A is given by

$$\mathbf{F} = \mathbf{F}_{AB} + \mathbf{F}_{AC}.$$

- Easily generalize to several charges.
- Experimental evidence verified this law to accuracy of 10^{-4}.
- Very accurate even at atomic distances and high strengths of forces.

Linear Superposition

Force $\mathbf{F}_{AB}$ on a charge, say A, due to another charge, say B, is independent of presence of a third charge, say C. Total force on A is given by

$$\mathbf{F} = \mathbf{F}_{AB} + \mathbf{F}_{AC}.$$

- Easily generalize to several charges.
- Experimental evidence verified this law to accuracy of 10^{-4}.
- Very accurate even at atomic distances and high strengths of forces.
- Non-linearities are evident at subatomic level and are legitimately incorporated in quantum theory.

Linear Superposition

Force $\mathbf{F}_{AB}$ on a charge, say A, due to another charge, say B, is independent of presence of a third charge, say C. Total force on A is given by

$$\mathbf{F} = \mathbf{F}_{AB} + \mathbf{F}_{AC}.$$

- Easily generalize to several charges.
- Experimental evidence verified this law to accuracy of 10^{-4}.
- Very accurate even at atomic distances and high strengths of forces.
- Non-linearities are evident at subatomic level and are legitimately incorporated in quantum theory.

Classical Electrodynamic Theory is built on this principle.

Electric Field

If there are several point charges, q_i; $i = 1, \ldots, n$, at locations $\mathbf{r}_i$, then electric field at $\mathbf{r}$ is defined as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i\ (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}$$

- Electric field is a vector quantity.
- Linear superposition holds for electric field.
- If a point charge of magnitude Q, is kept at $\mathbf{r}$, then the net force on the charge
$$\mathbf{F} = Q\mathbf{E}(\mathbf{r}).$$

Electric Field

If there are several point charges, q_i; $i = 1, \dots, n$, at locations $\mathbf{r}_i$, then electric field at $\mathbf{r}$ is defined as

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \frac{q_i \ (\mathbf{r} - \mathbf{r}_i)}{|\mathbf{r} - \mathbf{r}_i|^3}$$

- Electric field is a vector quantity.
- Linear superposition holds for electric field.
- If a point charge of magnitude Q, is kept at $\mathbf{r}$, then the net force on the charge

$$\mathbf{F} = Q\mathbf{E}(\mathbf{r}).$$

- Is electric field, a real physical quantity?

Electric Field

If there is continuous charge distribution with volume charge density ρ then electric field at $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}^{'}) \left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}'|^3} dv^{'}.$$

Electric Field

If there is continuous charge distribution with volume charge density ρ then electric field at $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}^{'}) \left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}'|^3} dv^{'}.$$

Clearly, if there is only surface charge with density σ, the definition would reduce to

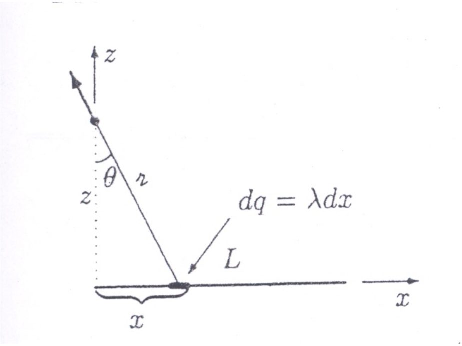
$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\mathbf{r}^{'}) \left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}'|^3} dS^{'}.$$

Electric Field

Example

[G2.3] Straight line segment $C : \mathbf{r}^{'}(t) = (t, 0, 0)$; $t \in [0, L]$ with uniform linear charge density λ_0. Calculate electric field at $(0, 0, z)$. Electric Field

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_C \frac{\lambda(\mathbf{r}^{'}) \left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}^{'}|^3} dl^{'}$$



Electric Field

Example

- $\mathbf{r} = (0, 0, z)$, $\mathbf{r}^{'} = (t, 0, 0)$

Electric Field

Example

- $\mathbf{r} = (0, 0, z)$, $\mathbf{r}^{'} = (t, 0, 0)$
- $\mathbf{r} - \mathbf{r}^{'}(t) = (-t, 0, z)$,

Electric Field

Example

- $\mathbf{r} = (0, 0, z)$, $\mathbf{r}^{'} = (t, 0, 0)$
- $\mathbf{r} - \mathbf{r}^{'}(t) = (-t, 0, z)$,
- $\left|\mathbf{r} - \mathbf{r}^{'}(t)\right| = \sqrt{t^2 + z^2}$

Electric Field

Example

- $\mathbf{r} = (0, 0, z)$, $\mathbf{r}^{'} = (t, 0, 0)$
- $\mathbf{r} - \mathbf{r}^{'}(t) = (-t, 0, z)$,
- $\left|\mathbf{r} - \mathbf{r}^{'}(t)\right| = \sqrt{t^2 + z^2}$
- $\lambda(\mathbf{r}^{'}(t)) = \lambda_0$

Electric Field

Example

- $\mathbf{r} = (0, 0, z)$, $\mathbf{r}^{'} = (t, 0, 0)$
- $\mathbf{r} - \mathbf{r}^{'}(t) = (-t, 0, z)$,
- $\left|\mathbf{r} - \mathbf{r}^{'}(t)\right| = \sqrt{t^2 + z^2}$
- $\lambda(\mathbf{r}^{'}(t)) = \lambda_0$
- $dl' = dt$

Electric Field

Example

- $\mathbf{r} = (0, 0, z)$, $\mathbf{r}' = (t, 0, 0)$
- $\mathbf{r} - \mathbf{r}'(t) = (-t, 0, z)$,
- $\left|\mathbf{r} - \mathbf{r}'(t)\right| = \sqrt{t^2 + z^2}$
- $\lambda(\mathbf{r}'(t)) = \lambda_0$
- $dl' = dt$

Then,

$$\mathbf{E}(\mathbf{r}) = \frac{\lambda_0}{4\pi\epsilon_0} \int_0^L \frac{(-t\hat{\mathbf{x}} + z\hat{\mathbf{z}})}{(t^2 + z^2)^{3/2}} dt$$

Electric Field

Example

- $\mathbf{r} = (0, 0, z)$, $\mathbf{r}^{'} = (t, 0, 0)$
- $\mathbf{r} - \mathbf{r}^{'}(t) = (-t, 0, z)$,
- $\left|\mathbf{r} - \mathbf{r}^{'}(t)\right| = \sqrt{t^2 + z^2}$
- $\lambda(\mathbf{r}^{'}(t)) = \lambda_0$
- $dl' = dt$

Then,

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{\lambda_0}{4\pi\epsilon_0} \int_0^L \frac{(-t\hat{\mathbf{x}} + z\hat{\mathbf{z}})}{(t^2 + z^2)^{3/2}} dt \\
&= \frac{\lambda_0}{4\pi\epsilon_0 z} \left[\left(-1 + \frac{z}{\sqrt{z^2 + L^2}} \right) \hat{\mathbf{x}} + \left(\frac{L}{\sqrt{z^2 + L^2}} \right) \hat{\mathbf{z}} \right]
\end{aligned}$$

Electric Field

Example

- $\mathbf{r} = (0, 0, z)$, $\mathbf{r}^{'} = (t, 0, 0)$
- $\mathbf{r} - \mathbf{r}^{'}(t) = (-t, 0, z)$,
- $\left|\mathbf{r} - \mathbf{r}^{'}(t)\right| = \sqrt{t^2 + z^2}$
- $\lambda(\mathbf{r}^{'}(t)) = \lambda_0$
- $dl' = dt$

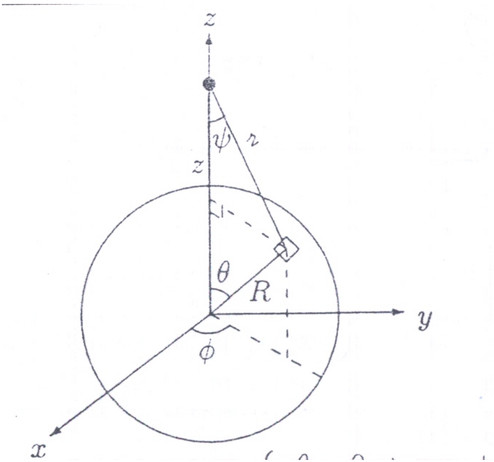
Then,

$$\begin{aligned} \mathbf{E}(\mathbf{r}) &= \frac{\lambda_0}{4\pi\epsilon_0} \int_0^L \frac{(-t\hat{\mathbf{x}} + z\hat{\mathbf{z}})}{(t^2 + z^2)^{3/2}} dt \\ &= \frac{\lambda_0}{4\pi\epsilon_0 z} \left[\left(-1 + \frac{z}{\sqrt{z^2 + L^2}} \right) \hat{\mathbf{x}} + \left(\frac{L}{\sqrt{z^2 + L^2}} \right) \hat{\mathbf{z}} \right] \\ &\approx \frac{\lambda_0}{4\pi\epsilon_0 z} \left[-\frac{L^2}{2z^2} \hat{\mathbf{x}} + \frac{L}{z} \hat{\mathbf{z}} \right] \quad z \gg L \end{aligned}$$

Electric Field

Example

[G2.7] Spherical surface of Radius R with uniform charge density $\sigma_0 = q/4\pi R^2$. Calculate Electric field at $\mathbf{r} = (0, 0, z)$.



Electric Field

- Target Point $\mathbf{r} = (0, 0, z)$, Source Point coordinates $(R, \theta^{'}, \phi^{'})$.

Electric Field

- Target Point $\mathbf{r} = (0, 0, z)$, Source Point coordinates $(R, \theta^{'}, \phi^{'})$.
- Vector $\mathbf{r}^{'} = R\hat{\mathbf{r}}^{'} = R\sin\theta^{'}\cos\phi^{'}\hat{\mathbf{x}} + R\sin\theta^{'}\sin\phi^{'}\hat{\mathbf{y}} + R\cos\theta^{'}\hat{\mathbf{z}}$

Electric Field

- Target Point $\mathbf{r} = (0, 0, z)$, Source Point coordinates $(R, \theta^{'}, \phi^{'})$.
- Vector $\mathbf{r}^{'} = R\hat{\mathbf{r}}^{'} = R\sin\theta^{'}\cos\phi^{'}\hat{\mathbf{x}} + R\sin\theta^{'}\sin\phi^{'}\hat{\mathbf{y}} + R\cos\theta^{'}\hat{\mathbf{z}}$
- $\mathbf{r} - \mathbf{r}^{'} = -R\sin\theta^{'}\cos\phi^{'}\hat{\mathbf{x}} - R\sin\theta^{'}\sin\phi^{'}\hat{\mathbf{y}} + \left(z - R\cos\theta^{'}\right)\hat{\mathbf{z}}$

Electric Field

- Target Point $\mathbf{r} = (0, 0, z)$, Source Point coordinates $(R, \theta^{'}, \phi^{'})$.
- Vector $\mathbf{r}^{'} = R\hat{\mathbf{r}}^{'} = R \sin\theta^{'} \cos\phi^{'} \hat{\mathbf{x}} + R \sin\theta^{'} \sin\phi^{'} \hat{\mathbf{y}} + R \cos\theta^{'} \hat{\mathbf{z}}$
- $\mathbf{r} - \mathbf{r}^{'} = -R \sin\theta^{'} \cos\phi^{'} \hat{\mathbf{x}} - R \sin\theta^{'} \sin\phi^{'} \hat{\mathbf{y}} + \left(z - R\cos\theta^{'}\right) \hat{\mathbf{z}}$
- $\left|\mathbf{r} - \mathbf{r}^{'}\right| = \sqrt{R^2 + z^2 - 2Rz\cos\theta^{'}}$

Electric Field

- Target Point $\mathbf{r} = (0, 0, z)$, Source Point coordinates $(R, \theta^{'}, \phi^{'})$.
- Vector $\mathbf{r}^{'} = R\hat{\mathbf{r}}^{'} = R \sin\theta^{'} \cos\phi^{'} \hat{\mathbf{x}} + R \sin\theta^{'} \sin\phi^{'} \hat{\mathbf{y}} + R \cos\theta^{'} \hat{\mathbf{z}}$
- $\mathbf{r} - \mathbf{r}^{'} = -R \sin\theta^{'} \cos\phi^{'} \hat{\mathbf{x}} - R \sin\theta^{'} \sin\phi^{'} \hat{\mathbf{y}} + \left(z - R\cos\theta^{'}\right) \hat{\mathbf{z}}$
- $\left|\mathbf{r} - \mathbf{r}^{'}\right| = \sqrt{R^2 + z^2 - 2Rz\cos\theta^{'}}$
- Elementary area (at $\mathbf{r}^{'}$) $dS^{'} = R^2 \sin\theta^{'} d\theta^{'} d\phi^{'}$

Electric Field

- Target Point $\mathbf{r} = (0, 0, z)$, Source Point coordinates $(R, \theta^{'}, \phi^{'})$.
- Vector $\mathbf{r}^{'} = R\hat{\mathbf{r}}^{'} = R\sin\theta^{'}\cos\phi^{'}\hat{\mathbf{x}} + R\sin\theta^{'}\sin\phi^{'}\hat{\mathbf{y}} + R\cos\theta^{'}\hat{\mathbf{z}}$
- $\mathbf{r} - \mathbf{r}^{'} = -R\sin\theta^{'}\cos\phi^{'}\hat{\mathbf{x}} - R\sin\theta^{'}\sin\phi^{'}\hat{\mathbf{y}} + \left(z - R\cos\theta^{'}\right)\hat{\mathbf{z}}$
- $\left|\mathbf{r} - \mathbf{r}^{'}\right| = \sqrt{R^2 + z^2 - 2Rz\cos\theta^{'}}$
- Elementary area (at $\mathbf{r}^{'}$) $dS^{'} = R^2\sin\theta^{'}\,d\theta^{'}\,d\phi^{'}$

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\mathbf{r}^{'})\left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}^{'}|^3}\,dS^{'}$$

Electric Field

- Target Point $\mathbf{r} = (0, 0, z)$, Source Point coordinates $(R, \theta^{'}, \phi^{'})$.
- Vector $\mathbf{r}^{'} = R\hat{\mathbf{r}}^{'} = R \sin\theta^{'} \cos\phi^{'} \hat{\mathbf{x}} + R \sin\theta^{'} \sin\phi^{'} \hat{\mathbf{y}} + R\cos\theta^{'} \hat{\mathbf{z}}$
- $\mathbf{r} - \mathbf{r}^{'} = -R \sin\theta^{'} \cos\phi^{'} \hat{\mathbf{x}} - R \sin\theta^{'} \sin\phi^{'} \hat{\mathbf{y}} + \left(z - R\cos\theta^{'}\right) \hat{\mathbf{z}}$
- $\left|\mathbf{r} - \mathbf{r}^{'}\right| = \sqrt{R^2 + z^2 - 2Rz\cos\theta^{'}}$
- Elementary area (at $\mathbf{r}^{'}$) $dS^{'} = R^2 \sin\theta^{'} d\theta^{'} d\phi^{'}$

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\mathbf{r}^{'}) \left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}^{'}|^3} dS^{'} \\
&= \frac{\sigma_0}{4\pi\epsilon_0} \int_0^{\pi} \int_0^{2\pi} \frac{R^2 \sin\theta^{'} d\theta^{'} d\phi^{'}}{(R^2 + z^2 - 2Rz\cos\theta^{'})^{3/2}} \\
&\quad \times [-R \sin\theta^{'} \cos\phi^{'} \hat{\mathbf{x}} - R \sin\theta^{'} \sin\phi^{'} \hat{\mathbf{y}} + \left(z - R\cos\theta^{'}\right) \hat{\mathbf{z}}]
\end{aligned}$$

Electric Field

- Target Point $\mathbf{r} = (0, 0, z)$, Source Point coordinates $(R, \theta^{'}, \phi^{'})$.
- Vector $\mathbf{r}^{'} = R\hat{\mathbf{r}}^{'} = R\sin\theta^{'}\cos\phi^{'}\hat{\mathbf{x}} + R\sin\theta^{'}\sin\phi^{'}\hat{\mathbf{y}} + R\cos\theta^{'}\hat{\mathbf{z}}$
- $\mathbf{r} - \mathbf{r}^{'} = -R\sin\theta^{'}\cos\phi^{'}\hat{\mathbf{x}} - R\sin\theta^{'}\sin\phi^{'}\hat{\mathbf{y}} + \left(z - R\cos\theta^{'}\right)\hat{\mathbf{z}}$
- $\left|\mathbf{r} - \mathbf{r}^{'}\right| = \sqrt{R^2 + z^2 - 2Rz\cos\theta'}$
- Elementary area (at $\mathbf{r}^{'}$) $dS^{'} = R^2\sin\theta^{'}\,d\theta^{'}\,d\phi^{'}$

$$\begin{aligned}
\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\mathbf{r}^{'})\left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}^{'}|^3}\,dS^{'} \\
&= \frac{\sigma_0}{4\pi\epsilon_0}\int_0^{\pi}\int_0^{2\pi}\frac{R^2\sin\theta^{'}\,d\theta^{'}\,d\phi^{'}}{(R^2 + z^2 - 2Rz\cos\theta')^{3/2}} \\
&\quad \times[-R\sin\theta^{'}\cos\phi^{'}\hat{\mathbf{x}} - R\sin\theta^{'}\sin\phi^{'}\hat{\mathbf{y}} + \left(z - R\cos\theta^{'}\right)\hat{\mathbf{z}}] \\
&= \frac{(4\pi R^2)\sigma_0}{4\pi\epsilon_0\, z^2}\,\hat{\mathbf{z}}
\end{aligned}$$

Curl of Electric Field

Suppose a point charge of magnitude q is placed at origin. Electric field at a point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

Now curl of electric field will be

$$\nabla \times \mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ (x/r^3) & (y/r^3) & (z/r^3) \end{vmatrix}$$

Curl of Electric Field

Suppose a point charge of magnitude q is placed at origin. Electric field at a point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

Now curl of electric field will be

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ (x/r^3) & (y/r^3) & (z/r^3) \end{vmatrix} \\ [\nabla \times \mathbf{E}(\mathbf{r})]_x &= \frac{q}{4\pi\epsilon_0}\left[\partial_y\left(\frac{z}{r^3}\right) - \partial_z\left(\frac{y}{r^3}\right)\right] \end{aligned}$$

Curl of Electric Field

Suppose a point charge of magnitude q is placed at origin. Electric field at a point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

Now curl of electric field will be

$$\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ (x/r^3) & (y/r^3) & (z/r^3) \end{vmatrix} \\
[\nabla \times \mathbf{E}(\mathbf{r})]_x &= \frac{q}{4\pi\epsilon_0}\left[\partial_y\left(\frac{z}{r^3}\right) - \partial_z\left(\frac{y}{r^3}\right)\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[\left(-\frac{3yz}{r^5}\right) - \partial_z\left(\frac{-3zy}{r^5}\right)\right] = 0
\end{aligned}$$

Curl of Electric Field

Suppose a point charge of magnitude q is placed at origin. Electric field at a point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

Now curl of electric field will be

$$\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\begin{vmatrix} \hat{\mathbf{x}} & \hat{\mathbf{y}} & \hat{\mathbf{z}} \\ \partial_x & \partial_y & \partial_z \\ (x/r^3) & (y/r^3) & (z/r^3) \end{vmatrix} \\
[\nabla \times \mathbf{E}(\mathbf{r})]_x &= \frac{q}{4\pi\epsilon_0}\left[\partial_y\left(\frac{z}{r^3}\right) - \partial_z\left(\frac{y}{r^3}\right)\right] \\
&= \frac{q}{4\pi\epsilon_0}\left[\left(-\frac{3yz}{r^5}\right) - \partial_z\left(\frac{-3zy}{r^5}\right)\right] = 0
\end{aligned}$$

Thus

$$\nabla \times \mathbf{E}(\mathbf{r}) = 0$$

Curl of Electric Field

Now we extend the result to arbitrary charge distribution ρ. Electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}^{'}) \left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}^{'}|^3} dv^{'}.$$

Then curl with respect to variable $\mathbf{r}$

$$\nabla \times \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \nabla \times \int \frac{\rho(\mathbf{r}^{'}) \left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}^{'}|^3} dv^{'}$$

Curl of Electric Field

Now we extend the result to arbitrary charge distribution ρ. Electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}')\left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3} dv'.$$

Then curl with respect to variable $\mathbf{r}$

$$\begin{aligned}
\nabla \times \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \nabla \times \int \frac{\rho(\mathbf{r}')\left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3} dv' \\
&= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}') \left(\nabla \times \frac{\left(\mathbf{r} - \mathbf{r}'\right)}{|\mathbf{r} - \mathbf{r}'|^3} \right) dv' \\
&= 0
\end{aligned}$$

Curl of electric field is always zero!

Curl of Electric Field

Now we extend the result to arbitrary charge distribution ρ. Electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\mathbf{r}^{'}) \left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}^{'}|^3} dv^{'} .$$

Then curl with respect to variable $\mathbf{r}$

$$\begin{aligned} \nabla \times \mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0} \nabla \times \int \frac{\rho(\mathbf{r}^{'}) \left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}^{'}|^3} dv^{'} \\ &= \frac{1}{4\pi\epsilon_0} \int \rho(\mathbf{r}^{'}) \left(\nabla \times \frac{\left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}^{'}|^3} \right) dv^{'} \\ &= 0 \end{aligned}$$

Curl of electric field is always zero!

$$\left(\nabla \times \frac{\left(\mathbf{r} - \mathbf{r}^{'}\right)}{|\mathbf{r} - \mathbf{r}^{'}|^3} \right)_x = \left(-3(z - z')\frac{(y - y')}{|\mathbf{r} - \mathbf{r}'|^3} + -3(y - y')\frac{(z - z')}{|\mathbf{r} - \mathbf{r}'|^3} \right) = 0$$

Divergence of Electric Field

Suppose a point charge of magnitude q is placed at origin. Volume charge density is $\rho(\mathbf{r}) = q\delta^3(\mathbf{r})$. Electric field at a point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

Now divergence of electric field will be

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right)$$

Divergence of Electric Field

Remember from previous lecture:

$$\begin{aligned} \nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) &= 0 \quad \text{if } \mathbf{r} \neq 0. \\ \text{And } \int_V \left(\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) \right) dv &= 4\pi \end{aligned}$$

This just looks like definition of Dirac's Delta delta function! Clearly,

$$\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2} \right) = 4\pi\delta(x)\delta(y)\delta(z)$$

Then,

$$\begin{aligned} \int_V (4\pi\delta(x)\delta(y)\delta(z))\, dv &= 4\pi \int_{-\infty}^{\infty} \delta(x)\, dx \int_{-\infty}^{\infty} \delta(y)\, dy \int_{-\infty}^{\infty} \delta(z)\, dz \\ &= 4\pi. \end{aligned}$$

Remember: $\delta(x)\delta(y)\delta(z) = \delta^3(\mathbf{r})$ (3D Dirac delta function).

Divergence of Electric Field

Suppose a point charge of magnitude q is placed at origin. Electric field at a point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

Now divergence of electric field will be

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right)$$

Divergence of Electric Field

Suppose a point charge of magnitude q is placed at origin. Electric field at a point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

Now divergence of electric field will be

$$\begin{aligned} \nabla \cdot \mathbf{E}(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) \\ &= \frac{q}{4\pi\epsilon_0}\left(4\pi\delta^3(\mathbf{r})\right) \end{aligned}$$

Divergence of Electric Field

Suppose a point charge of magnitude q is placed at origin. Electric field at a point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

Now divergence of electric field will be

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) \\
&= \frac{q}{4\pi\epsilon_0}\left(4\pi\delta^3(\mathbf{r})\right) \\
&= \frac{q\delta^3(\mathbf{r})}{\epsilon_0}
\end{aligned}$$

Divergence of Electric Field

Suppose a point charge of magnitude q is placed at origin. Electric field at a point $\mathbf{r}$ is

$$\mathbf{E}(\mathbf{r}) = \frac{q}{4\pi\epsilon_0}\frac{\hat{\mathbf{r}}}{r^2}.$$

Now divergence of electric field will be

$$\begin{aligned}
\nabla \cdot \mathbf{E}(\mathbf{r}) &= \frac{q}{4\pi\epsilon_0}\nabla \cdot \left(\frac{\hat{\mathbf{r}}}{r^2}\right) \\
&= \frac{q}{4\pi\epsilon_0}\left(4\pi\delta^3(\mathbf{r})\right) \\
&= \frac{q\delta^3(\mathbf{r})}{\epsilon_0} \\
&= \frac{\rho(\mathbf{r})}{\epsilon_0}
\end{aligned}$$

Divergence of Electric Field

Now we extend the result to arbitrary charge distribution ρ. Electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}^{'})\left(\mathbf{r}-\mathbf{r}^{'}\right)}{|\mathbf{r}-\mathbf{r}^{'}|^3} dv^{'} .$$

Then divergence with respect to variable $\mathbf{r}$

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\nabla \cdot \int \frac{\rho(\mathbf{r}^{'})\left(\mathbf{r}-\mathbf{r}^{'}\right)}{|\mathbf{r}-\mathbf{r}^{'}|^3} dv^{'}$$

Divergence of Electric Field

Now we extend the result to arbitrary charge distribution ρ. Electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}^{'})\left(\mathbf{r}-\mathbf{r}^{'}\right)}{|\mathbf{r}-\mathbf{r}'|^3}dv^{'}.$$

Then divergence with respect to variable $\mathbf{r}$

$$\begin{aligned}\nabla\cdot\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\nabla\cdot\int \frac{\rho(\mathbf{r}^{'})\left(\mathbf{r}-\mathbf{r}^{'}\right)}{|\mathbf{r}-\mathbf{r}'|^3}dv^{'} \\ &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}^{'})\left(\nabla\cdot\frac{\left(\mathbf{r}-\mathbf{r}^{'}\right)}{|\mathbf{r}-\mathbf{r}'|^3}\right)dv^{'}\end{aligned}$$

Divergence of Electric Field

Now we extend the result to arbitrary charge distribution ρ. Electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}^{'})\left(\mathbf{r}-\mathbf{r}^{'}\right)}{|\mathbf{r}-\mathbf{r}^{'}|^3}dv^{'}.$$

Then divergence with respect to variable $\mathbf{r}$

$$\begin{aligned}\nabla\cdot\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\nabla\cdot\int \frac{\rho(\mathbf{r}^{'})\left(\mathbf{r}-\mathbf{r}^{'}\right)}{|\mathbf{r}-\mathbf{r}^{'}|^3}dv^{'} \\ &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}^{'})\left(\nabla\cdot\frac{\left(\mathbf{r}-\mathbf{r}^{'}\right)}{|\mathbf{r}-\mathbf{r}^{'}|^3}\right)dv^{'} \\ &= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}^{'})\left(4\pi\delta^3(\mathbf{r}-\mathbf{r}^{'})\right)dv^{'}\end{aligned}$$

Divergence of Electric Field

Now we extend the result to arbitrary charge distribution ρ. Electric field is given by

$$\mathbf{E}(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\mathbf{r}')\left(\mathbf{r}-\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|^3}dv'.$$

Then divergence with respect to variable $\mathbf{r}$

$$\begin{aligned}
\nabla\cdot\mathbf{E}(\mathbf{r}) &= \frac{1}{4\pi\epsilon_0}\nabla\cdot\int \frac{\rho(\mathbf{r}')\left(\mathbf{r}-\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|^3}dv' \\
&= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left(\nabla\cdot\frac{\left(\mathbf{r}-\mathbf{r}'\right)}{|\mathbf{r}-\mathbf{r}'|^3}\right)dv' \\
&= \frac{1}{4\pi\epsilon_0}\int \rho(\mathbf{r}')\left(4\pi\delta^3(\mathbf{r}-\mathbf{r}')\right)dv' \\
&= \frac{1}{\epsilon_0}\int \rho(\mathbf{r}')\,\delta(\mathbf{r}-\mathbf{r}')dv' = \frac{1}{\epsilon_0}\rho(\mathbf{r})
\end{aligned}$$

Gauss's Law

Gauss's Law

In the neighbourhood of a point $\mathbf{r}$, the charge density is given by $\rho(\mathbf{r})$ and the electric field by $\mathbf{E}(\mathbf{r})$, then

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}$$

This is also called differential form of Gauss's Law.

Gauss's Law

Electric field in space is given by

$$\mathbf{E}(\mathbf{r}) = Ae^{-\lambda r}(1+\lambda r)\frac{\hat{\mathbf{r}}}{r^2}$$

$$\rho(\mathbf{r}) = \epsilon_0 \nabla \cdot \mathbf{E}(\mathbf{r})$$

The divergence formula

$$\nabla \cdot \mathbf{E} \quad = \quad \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(E_\phi)$$

Gauss's Law

Electric field in space is given by

$$\mathbf{E}(\mathbf{r}) = Ae^{-\lambda r}(1+\lambda r)\frac{\hat{\mathbf{r}}}{r^2}$$

$$\rho(\mathbf{r}) = \epsilon_0 \nabla \cdot \mathbf{E}(\mathbf{r})$$

The divergence formula

$$\begin{aligned}
\nabla \cdot \mathbf{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta E_\theta) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(E_\phi) \\
&= \frac{1}{r^2}\frac{\partial}{\partial r}\left(Ae^{-\lambda r}(1+\lambda r)\right) = -A\frac{\lambda^2}{r}e^{-\lambda r}
\end{aligned}$$

Gauss's Law

Electric field in space is given by

$$\mathbf{E}(\mathbf{r}) = Ae^{-\lambda r}(1+\lambda r)\frac{\hat{\mathbf{r}}}{r^2}$$

$$\rho(\mathbf{r}) = \epsilon_0 \nabla \cdot \mathbf{E}(\mathbf{r})$$

The divergence formula

$$\begin{aligned}\nabla \cdot \mathbf{E} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta E_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\left(E_\phi\right) \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(Ae^{-\lambda r}(1+\lambda r)\right) = -A\frac{\lambda^2}{r}e^{-\lambda r}\end{aligned}$$

But there may be some point charge at origin since $|\mathbf{E}| \sim 1/r^2$ near origin. Integrate over a spherical surface of radius R

$$\begin{aligned}\int \mathbf{E}\cdot d\mathbf{S} &= 4\pi Ae^{-\lambda R}(1+\lambda R) \\ &\rightarrow 4\pi A \qquad \text{as } R \rightarrow 0\end{aligned}$$

Then

$$\rho(\mathbf{r}) = \epsilon_0 A\left(4\pi\delta^3(\mathbf{r}) - \frac{\lambda^2}{r}e^{-\lambda r}\right)$$

Electric Flux

Definition

Let S be a simple surface. Electric field in the region containing S is given by a vector field $\mathbf{E}$. The *flux of* $\mathbf{E}$ *through surface* S is defined as

$$\phi_S = \int_S \mathbf{E} \cdot d\mathbf{S}$$

Example

Suppose a point charge is kept at origin. Find the flux through a hemisphere of radius R centered at origin.

Consider elementary area dS at point $\mathbf{r} = R\hat{\mathbf{r}}$. $|\mathbf{r}| = R$ and $dS = R^2 \sin\theta d\theta d\phi$ and unit normal to dS is $\hat{\mathbf{r}}$. Flux is

$$\begin{aligned} \phi_S &= \int_S \mathbf{E} \cdot d\mathbf{S} \\ &= \frac{q}{4\pi\epsilon_0} \int_0^{\pi/2} \int_0^{2\pi} \left(\frac{\hat{\mathbf{r}}}{R^2}\right) \cdot \hat{\mathbf{r}}\, R^2 \sin\theta d\theta d\phi \\ &= \frac{q}{2\epsilon_0} \end{aligned}$$

Gauss's Law

Gauss's Law

Let **E** be the electric field defined on a volume V bounded by a closed surface S. Then the flux of **E** through the closed surface S is equal to the total charge in volume V.

The Gauss Law:

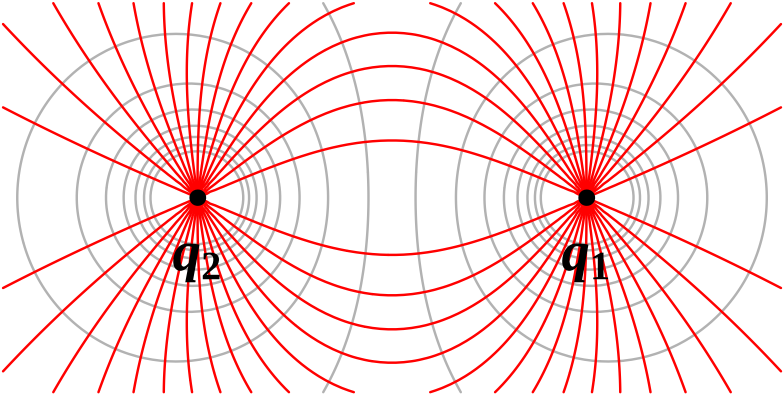
$$\begin{aligned} \nabla \cdot \mathbf{E}(\mathbf{r}) &= \frac{\rho(\mathbf{r})}{\epsilon_0} \\ \int_V \nabla \cdot \mathbf{E}(\mathbf{r})\, dv &= \int_V \frac{\rho(\mathbf{r})}{\epsilon_0}\, dv \\ \therefore \quad \oint_S \mathbf{E}(\mathbf{r}) \cdot d\mathbf{S} &= \frac{Q_{\text{enclosed}}}{\epsilon_0} \end{aligned}$$

where, Q_{enclosed} is the total charge in volume V.

This is known as integral form of Gauss's Law.

Gauss's Law

Integral form of Gauss Law can be interpreted in terms of field lines.



Electric field lines are shown in red and equi-potential lines in gray. Field lines are from $q_1 > 0$ to $q_2 = -q_1$. Flux through a surface is, then, number of lines crossing the surface.

Differential Equations for Electric Field

Here are two differential equations for electric field:

$$\nabla \cdot \mathbf{E}(\mathbf{r}) = \frac{\rho(\mathbf{r})}{\epsilon_0}$$
$$\nabla \times \mathbf{E}(\mathbf{r}) = 0$$

If ρ is given, can we find a unique solution for $\mathbf{E}$?

Theorem

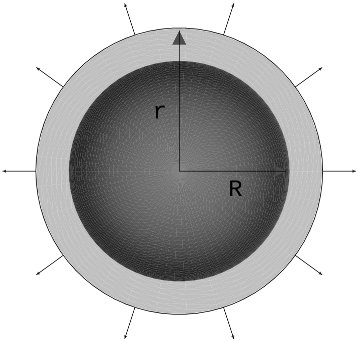
(Helmholtz Theorem) If ρ is nonzero on bounded volume, then there is a unique solution to the diff equations with $\mathbf{E} \longrightarrow 0$ as $\mathbf{r} \longrightarrow \infty$.

- Many problems are posed with different boundaries and boundary conditions
- However, this is rarely used to solve electrostatic problems.

Applications of Gauss's Law

Example

A uniformly charged sphere, with charge Q. Calculate Electric Field.



- A spherical surface of radius r (Guassian Surface).
- Magnitude of $\mathbf{E}$ on Gaussian surface is constant.
- Direction of $\mathbf{E}$ on Gaussian surface is known and is $\hat{\mathbf{r}}$.

Applications of Gauss's Law

$$\oint_{\text{Gaussian Surface}} \mathbf{E} \cdot d\mathbf{S} = |\mathbf{E}| \oint_{\text{Gaussian Surface}} \hat{\mathbf{r}} \cdot \hat{\mathbf{r}} dS$$
$$= |\mathbf{E}| \, 4\pi r^2$$

And this must be equal to Q/ϵ_0.

$$|\mathbf{E}| \, 4\pi r^2 = \frac{Q}{\epsilon_0}$$
$$\mathbf{E} = \frac{Q}{4\pi\epsilon_0 r^2}\hat{\mathbf{r}}$$