Physics II

Electromagnetism and Optics

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Product Rules (Gradient)

If f and g are scalar fields, so is fg. And if $\mathbf{A}$ and $\mathbf{B}$ are vector fields then $\mathbf{A} \cdot \mathbf{B}$ is a scalar field.

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- $\nabla(fg) = f\nabla g + g\nabla f$
- $\nabla(\mathbf{A} \cdot \mathbf{B}) = (\mathbf{A} \cdot \nabla)\mathbf{B} + (\mathbf{B} \cdot \nabla)\mathbf{A} + \mathbf{A} \times (\nabla \times \mathbf{B}) + \mathbf{A} \times (\nabla \times \mathbf{B})$

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Proof of the second Identity:

$$\begin{aligned}\nabla(\mathbf{A}\cdot\mathbf{B}) &= \nabla(A_xB_x + A_yB_y + A_zB_z)\\ &= (A_x\nabla B_x + A_y\nabla B_y + A_z\nabla B_z) + (B_x\nabla A_x + \cdots)\end{aligned}$$

The x-component of the first bracket:

$$\begin{array}{cccc} & +A_x\partial_xB_x & +A_y\partial_xB_y & +A_z\partial_xB_z\\ & +A_y\partial_yB_x & -A_y\partial_yB_x & \\ & +A_z\partial_zB_x & & -A_z\partial_zB_x\\ \hline (\mathbf{A}\cdot\nabla)B_x & & +A_y(\nabla\times\mathbf{B})_z & -A_z(\nabla\times\mathbf{B})_y\\ & (\mathbf{A}\cdot\nabla)B_x + (\mathbf{A}\times(\nabla\times\mathbf{B}))_x & & \end{array}$$

Product Rules (Divergence and Curl)

- $\nabla \cdot (f\mathbf{A}) = f(\nabla \cdot \mathbf{A}) + \mathbf{A} \cdot \nabla f$
- $\nabla \times (f\mathbf{A}) = f(\nabla \times \mathbf{A}) - \mathbf{A} \times \nabla f$
- $\nabla \cdot (\mathbf{A} \times \mathbf{B}) = \mathbf{B} \cdot (\nabla \times \mathbf{A}) - \mathbf{A} \cdot (\nabla \times \mathbf{B})$
- $\nabla \times (\mathbf{A} \times \mathbf{B}) = (\mathbf{B} \cdot \nabla)\mathbf{A} - (\mathbf{A} \cdot \nabla)\mathbf{B} + \mathbf{A} \cdot (\nabla \cdot \mathbf{B}) - \mathbf{B}(\nabla \cdot \mathbf{A})$

Higher Derivatives

Higher order derivatives of fields can be written in terms of partial derivatives like

$$\frac{\partial^2}{\partial x^2} \quad \frac{\partial^2}{\partial y^2} \quad \frac{\partial^2}{\partial x \partial y} \quad \frac{\partial^2}{\partial y \partial x}$$

The last two terms are called mixed partial derivatives.

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The last two terms are called mixed partial derivatives.

Theorem

If f has continuous second ordered partial derivatives, then the mixed partial derivatives are equal that is

$$\frac{\partial^2}{\partial x_j \partial x_k} f = \frac{\partial^2}{\partial x_k \partial x_j} f$$

where x_j and x_k are either x, y or z

Higher Derivatives

- Laplacian:

$$\nabla^2 f = \nabla \cdot (\nabla f) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) f$$

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$$\nabla \cdot (\nabla \times \mathbf{F}) = 0$$

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- Curl of a curl

$$\nabla \times (\nabla \times \mathbf{F}) = -\nabla^2 \mathbf{F} + \nabla(\nabla \cdot \mathbf{F})$$

Line (or Path) Integrals

Integration of real valued function of one variable,

$$\int_a^b f(x)dx$$

is quite familiar.

We want to extend this defintion to integrating scalar and vector fields over arbitrary paths.

How to describe paths?

Definition

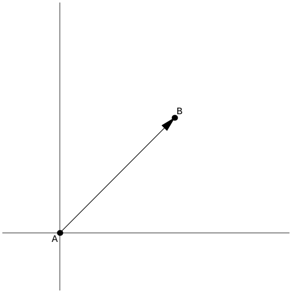
Let $[a, b] \in \mathbb{R}$ be a closed interval. A path in $\mathbb{R}^m$ is a continuous function $\mathbf{r} : [a.b] \to \mathbb{R}^m$. The path is called smooth if the derivative $\mathbf{r}'$ exists and is continuous.

When $m = 2$, the function $\mathbf{r}(t) = \hat{\mathbf{x}}x(t) + \hat{\mathbf{y}}y(t)$ can be split into two component functions.

Example

$$\left.\begin{array}{l} x(t) = t \\ y(t) = t \end{array}\right\} t \in [0, 1]$$

This is directed path.
Starting Point :$(0, 0)$ End Point: $(1, 1)$



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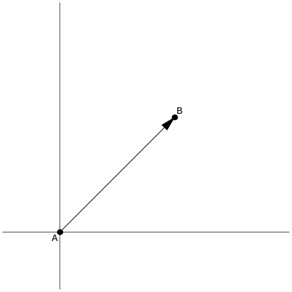
This is directed path.
Starting Point :$(0, 0)$ End Point: $(1, 1)$
Reversed path

$$\left.\begin{array}{l} x(t) = 1 - t \\ y(t) = 1 - t \end{array}\right\} t \in [0, 1]$$

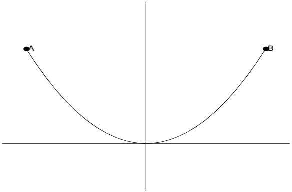


How to describe paths?

Example

Parabola

$$\left.\begin{array}{l} x(t) = t \\ y(t) = t^2 \end{array}\right\} t \in [-1, 1]$$



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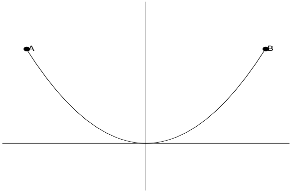
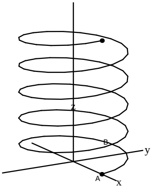


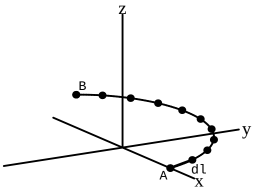
Helix

$$\left.\begin{aligned} x(t) &= \cos t \\ y(t) &= \sin t \\ z(t) &= t/2\pi \end{aligned}\right\} t \in [0, 10\pi]$$



Line Integral of Scalar Fields

- Split path into small segments of length dr
- Line Integral is limiting value of $\sum f \, dr$ over the path.



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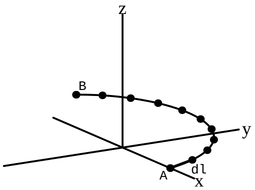

Definition

If $\mathbf{r}(t);\ t \in [a, b]$ is a path in $\mathbb{R}^m$ and f is scalar field over $\mathbb{R}^m$, then line integral of f over path $\mathbf{r}$ is

$$\int f\ dr = \int_a^b f(\mathbf{r}(t))\,|\mathbf{r}'(t)|\,dt.$$

For $m = 2$, $\mathbf{r}(t) = \hat{\mathbf{x}}x(t) + \hat{\mathbf{y}}y(t)$ and $|\mathbf{r}'(t)| = \left((dx/dt)^2 + (dy/dt)^2\right)^{1/2}$.

Line Integral of Scalar Fields

Example

Length of a circular arc

Let $\mathbf{r}(t) = (R\cos t, R\sin t);\ \ t \in [0, \alpha]$ then $\mathbf{r}'(t) = (-R\sin t, R\cos t)$ and $|\mathbf{r}'(t)| = R$

$$\begin{aligned} L &= \int_0^\alpha |\mathbf{r}'(t)| dt \\ &= \int_0^\alpha R dt = R\alpha \end{aligned}$$

Line Integral of Scalar Fields

Example

Centre of Mass

If $m(\mathbf{r})$ is linear mass density of a wire given by path $\mathbf{r}(t)\; t \in [a, b]$. Let $\mathbf{R}_{cm} = (x_{cm}, y_{cm})$. Then

$$\begin{aligned} x_{cm} &= \frac{1}{M} \int x(t) m(\mathbf{r}(t)) \left|\mathbf{r}'(t)\right| dt \\ y_{cm} &= \frac{1}{M} \int y(t) m(\mathbf{r}(t)) \left|\mathbf{r}'(t)\right| dt \end{aligned}$$

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\end{aligned}
$$

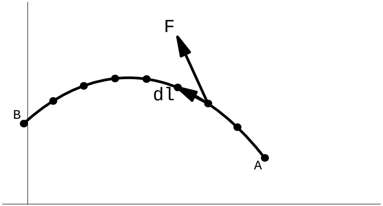
Or more compactly

$$
\mathbf{R}_{cm} = \frac{1}{M} \int \mathbf{r}(t) m(\mathbf{r}(t)) \left|\mathbf{r}'(t)\right| dt
$$

Thus line integral of a vector field can be defined in a straight-forward manner.

Line Integral of Vector Fields

To compute the work done by a force along the particle trajectory, we need to sum terms like $\sum \mathbf{F} \cdot d\mathbf{r}$



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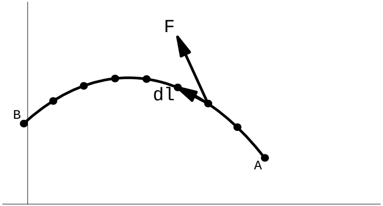

Definition

If $\mathbf{r}(t)$; $t \in [a, b]$ is a path in $\mathbb{R}^m$ and $\mathbf{F}$ is vector field over $\mathbb{R}^m$, then the line integral of $\mathbf{F}$ over path $\mathbf{r}$ is

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt.$$

Line Integral of Vector Fields

Example

A force field is given by $\alpha \mathbf{r}/r^3$ where $\alpha > 0$ is constant and $\mathbf{r}$ is position vector. Find the work done in moving a particle along a curve
$\mathbf{r}(t) = (\cos t, \sin t);\;\; t \in [0, 2\pi]$.

Given path is circular and closed (endpoint coincides with starting point). Along the path $r(t) = |\mathbf{r}(t)| = 1$.

$$\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \alpha \int_0^{2\pi} \frac{(\cos t, \sin t) \cdot (-\sin t, \cos t)}{r(t)^3} dt = 0$$

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Example

Another field: $\alpha\,(-\hat{\mathbf{x}}y + \hat{\mathbf{y}}x)$. With the same path as above

$$\int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t)\, dt = \alpha \int_0^{2\pi} (-\sin t, \cos t) \cdot (-\sin t, \cos t)\, dt = 2\pi\alpha$$

Surface Integrals

How to describe Surfaces

Common Form: $z = f(x, y)$ or $x = f(y, z)$ etc.

Examples

(a) $z = c$ is a plane parallel to XY plane

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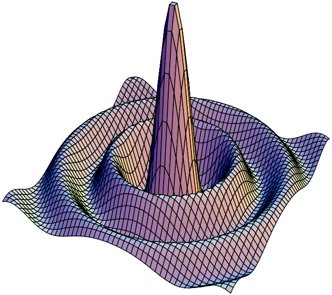
Examples

(a) $z = c$ is a plane parallel to XY plane

(b) $z = \sin\left(\sqrt{x^2 + y^2}\right) / \sqrt{x^2 + y^2}$



How to describe Surfaces

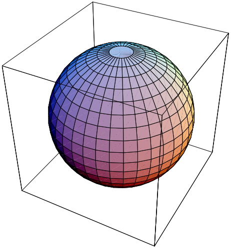
Another Form: $f(x, y, z) = c$

Examples

(a) Closed surfaces like sphere can be described by

$$z = \pm\sqrt{1 - x^2 - y^2}$$

and a more difficult boundary $x^2 + y^2 = 1$.



Alternatively one describe the entire sphere by

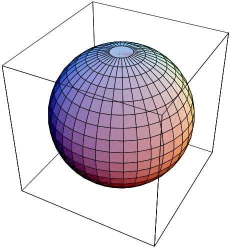
$$x^2 + y^2 + z^2 = 1$$

How to describe Surfaces

A general form: $S = \left\{ \mathbf{S}(u, v) = (x(u, v), y(u, v), z(u, v)) \,|\, (u, v) \in D \subset \mathbb{R}^2 \right\}$

Example

$\mathbf{S} = (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$ where $\phi \in [0, 2\pi]$ and $\theta \in [0, \pi]$



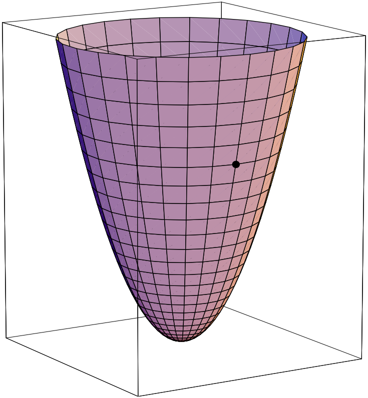
Parametric curves can be seen on the surface.

Elementary Area on a Surface

Consider a surface given by

$$\mathbf{S}(r,\phi) = \left(r\cos\phi, r\sin\phi, r^2\right)$$

with $r \in [0,3]$ and $\phi \in [0,2\pi]$
Let $\mathbf{A} = \mathbf{S}(2,0) = (2,0,4)$

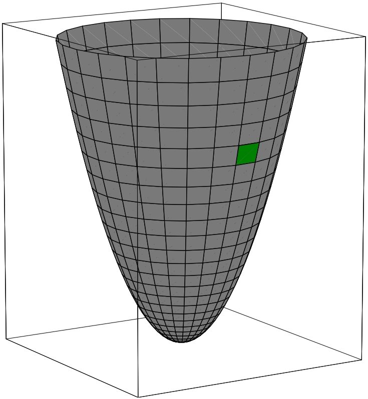


Elementary Area on a Surface

Consider a surface given by

$$\mathbf{S}(r, \phi) = \left(r\cos\phi, r\sin\phi, r^2\right)$$

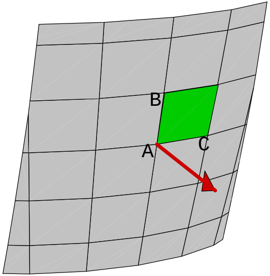
with $r \in [0, 3]$ and $\phi \in [0, 2\pi]$
Let $\mathbf{A} = \mathbf{S}(r = 2, \phi = 0) = (2, 0, 4)$
Elementary area is shown in green.



Elementary Area on a Surface

$$\mathbf{S}(r,\phi) = \left(r\cos\phi, r\sin\phi, r^2\right)$$

Let $\mathbf{A} = \mathbf{S}(2,0) = (2,0,4)$
Elementary area is shown in green.
$\overrightarrow{AB} = (\partial\mathbf{S}/\partial r)\; dr$ and
$\overrightarrow{AC} = (\partial\mathbf{S}/\partial\phi)\; d\phi$
Normal: $\mathbf{n} = (\partial\mathbf{S}/\partial r) \times (\partial\mathbf{S}/\partial\phi)$
Area: $dS = \left|\overrightarrow{AB} \times \overrightarrow{AC}\right| = |\mathbf{n}|\; dr\, d\phi$
Vector Area: $d\mathbf{S} = \hat{\mathbf{n}}\; dS$



Surface Integrals: Scalar Fields

Definition

Let S be a surface parametrized by $\mathbf{S}(u, v)$ with $(u, v) \in D$. Surface integral of a scalar field f is defined by

$$\int_S f\, dS = \int_D f\left(\mathbf{S}(u, v)\right)\, dS = \int_D f\left(\mathbf{S}(u, v)\right) \left|\frac{\partial \mathbf{S}}{\partial u} \times \frac{\partial \mathbf{S}}{\partial v}\right| du\, dv$$

Surface Integrals: Scalar Fields

Example

Area of a hemisphere:

- $\mathbf{S} = (R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$ with $\theta \in [0, \pi/2]$ and $\phi \in [0, 2\pi]$
- $\partial\mathbf{S}/\partial\theta = (R\cos\theta\cos\phi, R\cos\theta\sin\phi, -R\sin\theta)$
- $\partial\mathbf{S}/\partial\phi = (-R\sin\theta\sin\phi, R\sin\theta\cos\phi, 0)$
- $\partial\mathbf{S}/\partial\theta \times \partial\mathbf{S}/\partial\phi = R^2\left(\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta\right)$
- $|\partial\mathbf{S}/\partial\theta \times \partial\mathbf{S}/\partial\phi| = R^2\sin\theta$
- The area of hemisphere

$$\begin{aligned}\int_S dS &= \int_0^{\pi/2}\int_0^{2\pi} R^2\sin\theta\, d\theta\, d\phi \\ &= 2\pi R^2\end{aligned}$$

- How about $\mathbf{S} = (x, y, \sqrt{R^2 - x^2 - y^2})$ with $x^2 + y^2 \leq R^2$? Complete at home.

Surface Integrals: Vector Fields

Consider a steady state flow of incompressible fluid, which can be described by a velocity field $\mathbf{v}(\mathbf{r})$. Amount of fluid passing through a surface of area dS is given by $\mathbf{v} \cdot \hat{\mathbf{n}} dS$. This motivates the defintion of surface integrals of vector fields

Definition

Let S be a surface parametrized by $\mathbf{S}(u, v)$ with $(u, v) \in D$. Surface integral of a vector field $\mathbf{F}$ is defined by

$$\int_S \mathbf{F} \cdot d\mathbf{S} = \int_D \mathbf{F}\left(\mathbf{S}(u,v)\right) \cdot \hat{\mathbf{n}}\, dS = \int_D f\left(\mathbf{S}(u,v)\right) \mathbf{n}\, du\, dv$$

where $\mathbf{n} = \frac{\partial \mathbf{S}}{\partial \boldsymbol{u}} \times \frac{\partial \mathbf{S}}{\partial v}$

Surface Integrals: Vector Fields

Example

Let $\mathbf{F}(\mathbf{r}) = \hat{\mathbf{r}}/r^2 = (x, y, z)/(x^2 + y^2 + z^2)^{3/2}$

- $\mathbf{S} = (R\sin\theta\cos\phi, R\sin\theta\sin\phi, R\cos\theta)$ with $\theta \in [0, \pi]$ and $\phi \in [0, 2\pi]$
- $\partial\mathbf{S}/\partial\theta = (R\cos\theta\cos\phi, R\cos\theta\sin\phi, -R\sin\theta)$
- $\partial\mathbf{S}/\partial\phi = (-R\sin\theta\sin\phi, R\sin\theta\cos\phi, 0)$
- $\partial\mathbf{S}/\partial\theta \times \partial\mathbf{S}/\partial\phi = R^2\left(\sin^2\theta\cos\phi, \sin^2\theta\sin\phi, \sin\theta\cos\theta\right) = R^2\sin\theta\hat{\mathbf{n}} = R^2\sin\theta\hat{\mathbf{S}}$
- $|\partial\mathbf{S}/\partial\theta \times \partial\mathbf{S}/\partial\phi| = R^2\sin\theta$
- Surface integral is

$$\begin{aligned}\int_S \mathbf{F}\cdot\hat{\mathbf{n}}dS &= \int_0^{\pi/2}\int_0^{2\pi}\left(\frac{\hat{\mathbf{S}}}{|\mathbf{S}|^2}\right)\cdot\left(\hat{\mathbf{S}}R^2\sin\theta\right)\,d\theta\,d\phi \\ &= \int_0^{\pi/2}\int_0^{2\pi}\sin\theta\,d\theta\,d\phi = 4\pi.\end{aligned}$$

Volume Integrals

A volume in 3D is simply a region in $\mathbb{R}^3$. Thus volume integral of a scalar field f over a volume V is defined as

$$\int_V f(x, y, z)\, dx\, dy\, dz$$

Fundamental Theorem For Gradients

In some sense, this theorem says, integration is an inverse of differentiation.

Theorem

If ϕ is a differentiable scalar field with continuous gradient $\nabla\phi$ on open connected set S in $\mathbb{R}^3$ and $a, b \in S$, then

$$\int_a^b \nabla\phi \cdot d\mathbf{r} = \phi(b) - \phi(a)$$

over any smooth path joining a and b.

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over any smooth path joining a and b.

- $\int_a^b \nabla\phi \cdot d\mathbf{r}$ is independent of path, which is not the case ordinarily
- $\oint \nabla\phi \cdot d\mathbf{r} = 0$

Fundamental Theorem For Gradients

Another fundamental theorem:

Theorem

Let $\mathbf{F}$ *be a vector field over* $\mathbb{R}^3$ *such that its path integral between two points is independent of path. Define a scalar field* ϕ *such that*

$$\phi(\mathbf{r}) = \int_{\mathbf{a}}^{\mathbf{r}} \mathbf{F} \cdot d\mathbf{r}$$

where $\mathbf{a}$ *is some fixed point. Then* $\nabla\phi = \mathbf{F}$*.*

- True if $\oint \mathbf{F} \cdot d\mathbf{r} = 0$ for all closed paths (loops)

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Since $\nabla \times (\nabla\phi) = 0$, for $\mathbf{F}$ to be equal to gradient of some potential field, atlest $\nabla \times \mathbf{F} = 0$. Is this condition sufficient? Yes if it is true over convex sets.

Fundamental Theorem For Gradients

Check if $\mathbf{F} = yz\hat{\mathbf{x}} + xz\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$ can be written as a gradient of a potential?

Fundamental Theorem For Gradients

Check if $\mathbf{F} = yz\hat{\mathbf{x}} + xz\hat{\mathbf{y}} + xy\hat{\mathbf{z}}$ can be written as a gradient of a potential?

if $\phi(x, y, z) = xyz + c$ then $\nabla\phi = \mathbf{F}$

Fundamental Theorem For Divergence

This is simillar to the previous theorem.

Theorem

(Gauss Theorem) Let V be a solid region in $\mathbb{R}^3$ bounded by closed surface S. If $\mathbf{F}$ *is continuously differentiable on V then*

$$\int_V (\nabla \cdot \mathbf{F})\, dv = \oint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS$$

with $\hat{\mathbf{n}}$ is outer normal to S.

Inverse Square Field

Example

Let $\mathbf{F} = \hat{\mathbf{r}}/r^2$.

- $\nabla \cdot \mathbf{F}(\mathbf{r}) = 0$ if $\mathbf{r} \neq 0$.
- $\oint_S \mathbf{F} \cdot \hat{\mathbf{n}} dS = 4\pi$ if S is spherical surface centered at $\mathbf{r} = 0$.
- by Gauss theorem
$$\int_V (\nabla \cdot \mathbf{F})\, dv = 4\pi$$
- Contribution to right hand side integral is only at origin. Then $\nabla \cdot \mathbf{F}(0) = \infty$

Fundamental Theorem For Curl

Theorem

(Gauss Theorem) Let S be a smooth surface in $\mathbb{R}^3$ bounded by closed curve Γ. If $\mathbf{F}$ is continuously differentiable vector field, then

$$\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = \oint_\Gamma \mathbf{F} \cdot d\mathbf{r}$$

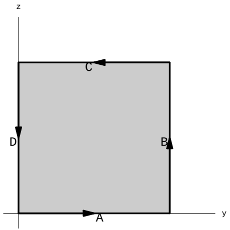
where direction of $d\mathbf{S}$ vector is determined by the right hand rule.

- $\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$ does not depend on the surface but only the boundary.
- $\oint_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 0$

Fundamental Theorem For Curl

Example

Let $\mathbf{F} = (2xz + 3y^2)\hat{\mathbf{y}} + 4yz^2\hat{\mathbf{z}}$ and surface S be square plane in yz plane given by $\mathbf{S} = (0, y, z)$ with $0 \leq y, z \leq 1$.

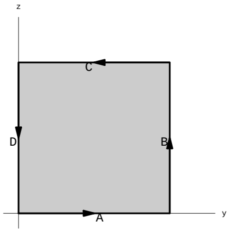


Fundamental Theorem For Curl

Example

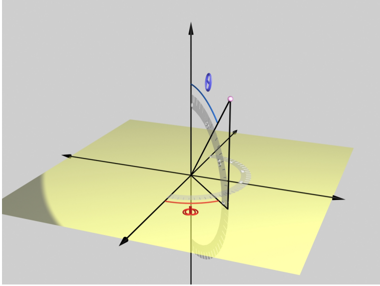
Let $\mathbf{F} = (2xz + 3y^2)\hat{\mathbf{y}} + 4yz^2\hat{\mathbf{z}}$ and surface S be square plane in yz plane given by $\mathbf{S} = (0, y, z)$ with $0 \leq y, z \leq 1$.

- In Figure: $d\mathbf{S} = dydz\hat{\mathbf{x}}$
- $\nabla \times \mathbf{F} = (4z^2 - 2x)\hat{\mathbf{x}} + 2z\hat{\mathbf{z}}$
- $\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S} = 4/3$
- $\int_A \mathbf{F} \cdot d\mathbf{r} = 1$, $\int_B \mathbf{F} \cdot d\mathbf{r} = 4/3$, $\int_C \mathbf{F} \cdot d\mathbf{r} = -1$, $\int_D \mathbf{F} \cdot d\mathbf{r} = 0$
- Stokes Theorem verified.



Spherical Coordinate System

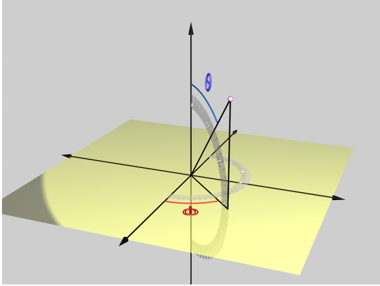
- position vector of P: $\mathbf{r}$
- Cartesian coordinates: (x, y, z).
- length of $\mathbf{r}$: $r = |\mathbf{r}|$
- projection of $\mathbf{r}$ onto XY plane: OQ
- angle between z-axis and $\mathbf{r}$: θ
- angle between x-axis and OQ: ϕ
- θ: zenith angle
- ϕ: azimuthal angle.
- spherical polar coordinates: ordered triplet (r, θ, ϕ)



Spherical Coordinate System

Spherical Coordinate System

$$
\begin{aligned}
r &= \sqrt{x^2 + y^2 + z^2} \\
\theta &= \tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) \\
\phi &= \tan^{-1}\left(\frac{y}{x}\right).
\end{aligned}
$$



Spherical Coordinate System

$$r = \sqrt{x^2 + y^2 + z^2}$$

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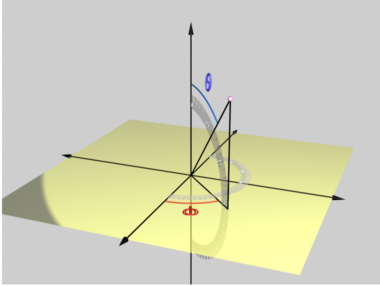


$$x = r\sin\theta\cos\phi$$

$$y = r\sin\theta\sin\phi$$

$$z = r\cos\theta$$

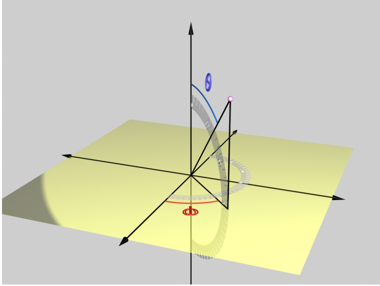
Spherical Coordinate System

Ranges of Spherical polar coordinates:
Clearly, $x, y, z \in (-\infty, \infty)$.

- $r \in [0, \infty)$,
- $\theta \in [0, \pi]$
- $\phi \in [0, 2\pi)$.

Note:

- ϕ is undefined for points on z-axis
- both θ and ϕ are undefined for the origin.



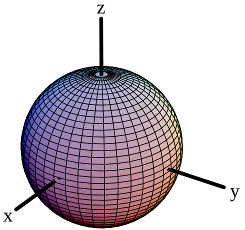
Spherical Coordinate System

Coordinate Surfaces can be obtained by keeping one of the coordinates constant.

$r =$ constant gives a spherical surface. Let $c > 0$.

$S = \{(c, \theta, \phi) \,|\, \theta \in [0, \pi], \phi \in [0, 2\pi)\}$

is a sphere of radius c.



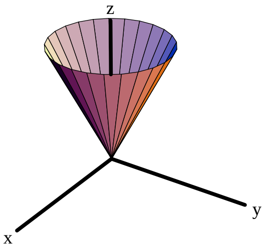
Spherical Coordinate System

Coordinate Surfaces can be Obtained by keeping one of the coordinates constant.

$\theta =$ constant gives a conical surface.
Let $c > 0$.

$$S = \{(r, c, \phi) \mid r \in [0, \infty], \phi \in [0, 2\pi)\}$$

is a cone of angle c.



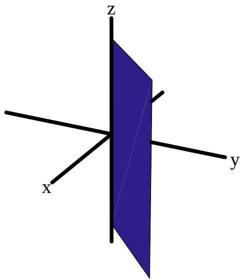
Spherical Coordinate System

Coordinate Surfaces can be Obtained by keeping one of the coordinates constant.

$\phi =$ constant gives a planar surface.
Let $c > 0$.

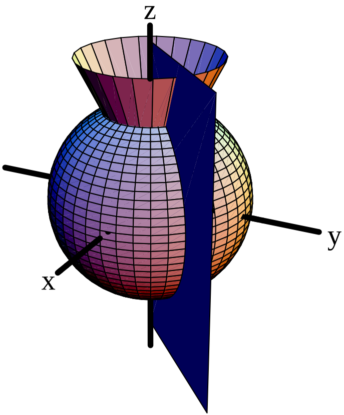
$$S = \{(r, \theta, c) \,|\, \theta \in [0, \pi], r \in [0, \infty)\}$$

is a half plane.



Spherical Coordinate System

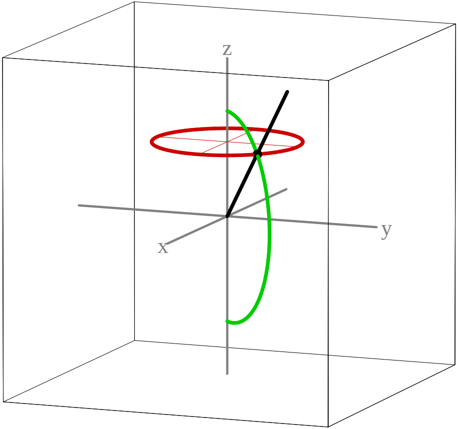
All surfaces:



Spherical Coordinate System

Coordinate Curves: Keeping two coordinates fixed, we get a path. Let $P = (r_0, \theta_0, \phi_0)$

- $\mathbf{S}_r = r\left(\sin\theta_0 \cos\phi_0, \sin\theta_0 \sin\phi_0, \cos\theta_0\right)$
- $\mathbf{S}_\theta = r_0\left(\sin\theta \cos\phi_0, \sin\theta \sin\phi_0, \cos\theta\right)$
- $\mathbf{S}_\phi = r_0\left(\sin\theta_0 \cos\phi, \sin\theta_0 \sin\phi, \cos\theta_0\right)$



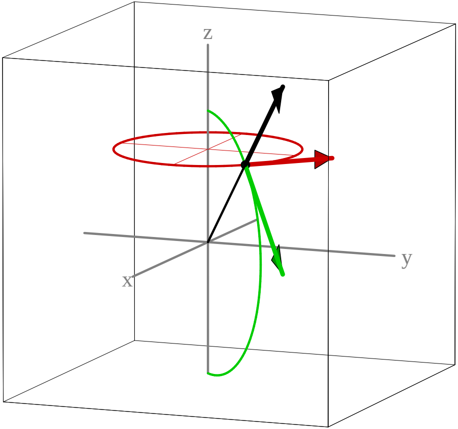
Spherical Coordinate System

Coordinate unit vectors are unit tangent vectors to coordinate curves at a given point. Let $P = (r_0, \theta_0, \phi_0)$. If $\mathbf{S} = r(\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)$

$$\hat{\mathbf{r}} = \frac{\partial \mathbf{S}}{\partial r} \Big/ \left| \frac{\partial \mathbf{S}}{\partial r} \right|$$

$$\hat{\theta} = \frac{\partial \mathbf{S}}{\partial \theta} \Big/ \left| \frac{\partial \mathbf{S}}{\partial \theta} \right|$$

$$\hat{\phi} = \frac{\partial \mathbf{S}}{\partial \phi} \Big/ \left| \frac{\partial \mathbf{S}}{\partial \phi} \right|$$



Spherical Coordinate System

Given, $\mathbf{S} = r\sin\theta\cos\phi\hat{\mathbf{x}} + r\sin\theta\sin\phi\hat{\mathbf{y}} + r\cos\theta\hat{\mathbf{z}}$, the Unit Vectors:

$$
\begin{aligned}
\hat{\mathbf{r}}(\theta,\phi) &= \frac{\partial \mathbf{S}}{\partial r}\Big/\left|\frac{\partial \mathbf{S}}{\partial r}\right| = \sin\theta\cos\phi\hat{\mathbf{x}} + \sin\theta\sin\phi\hat{\mathbf{y}} + \cos\theta\hat{\mathbf{z}} \\
\hat{\theta}(\theta,\phi) &= \frac{\partial \mathbf{S}}{\partial \theta}\Big/\left|\frac{\partial \mathbf{S}}{\partial \theta}\right| = \cos\theta\cos\phi\hat{\mathbf{x}} + \cos\theta\sin\phi\hat{\mathbf{y}} - \sin\theta\hat{\mathbf{z}} \\
\hat{\phi}(\theta,\phi) &= \frac{\partial \mathbf{S}}{\partial \phi}\Big/\left|\frac{\partial \mathbf{S}}{\partial \phi}\right| = -\sin\phi\hat{\mathbf{x}} + \cos\phi\hat{\mathbf{y}}
\end{aligned}
$$

Unit Vectors depend on the location on θ and ϕ.
Inverse Transformations are

$$
\begin{aligned}
\hat{\mathbf{x}} &= \sin\theta\cos\phi\hat{\mathbf{r}} + \cos\theta\cos\phi\hat{\theta} - \sin\phi\hat{\phi} \\
\hat{\mathbf{y}} &= \sin\theta\sin\phi\hat{\mathbf{r}} + \cos\theta\sin\phi\hat{\theta} + \cos\phi\hat{\phi} \\
\hat{\mathbf{z}} &= \cos\theta\hat{\mathbf{r}} - \sin\theta\hat{\theta}
\end{aligned}
$$

Spherical Coordinate System

Usually, the unit vectors are written without reference to the location, but it is understood by context.

Point	Cartesian	Spherical
P	$(1,0,0)$	$(1,\pi/2,0)$
Q	$(0,1,0)$	$(1,\pi/2,\pi/2)$

Now,

$\hat{\mathbf{r}}(P) = \hat{\mathbf{r}}(\theta = \pi/2, \phi = 0) = \hat{\mathbf{x}}$ and

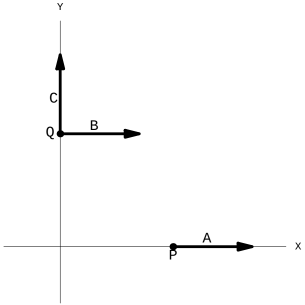
$\hat{\phi}(P) = \hat{\phi}(\theta = \pi/2, \phi = 0) = \hat{\mathbf{y}}$

But

$\hat{\mathbf{r}}(Q) = \hat{\mathbf{r}}(\theta = \pi/2, \phi = \pi/2) = \hat{\mathbf{y}}$ and

$\hat{\phi}(Q) = \hat{\phi}(\theta = \pi/2, \phi = \pi/2) = -\hat{\mathbf{x}}$

Vector	Cartesian	Spherical
A	$\hat{\mathbf{x}}$	$\hat{\mathbf{r}}$
B	$\hat{\mathbf{x}}$	$-\hat{\phi}$
C	$\hat{\mathbf{y}}$	$\hat{\mathbf{r}}$



Spherical Coordinate System

Cartesian unit vectors are constants and do not depend on position, but spherical unit vectors do!

$$\begin{aligned}
\frac{\partial}{\partial\theta}\hat{\mathbf{r}} &= \hat{\theta} \quad \frac{\partial}{\partial\phi}\hat{\mathbf{r}} = \sin\theta\hat{\phi} \\
\frac{\partial}{\partial\theta}\hat{\theta} &= -\hat{\mathbf{r}} \quad \frac{\partial}{\partial\phi}\hat{\theta} = \cos\theta\hat{\phi} \\
\frac{\partial}{\partial\theta}\hat{\phi} &= 0 \quad \frac{\partial}{\partial\phi}\hat{\phi} = -\sin\theta\hat{\mathbf{r}} - \cos\theta\hat{\theta}
\end{aligned}$$

Spherical Coordinate System

Position vector to any point $P = (r, \theta, \phi)$ is

$$\overrightarrow{OP} = \mathbf{r} = r\hat{\mathbf{r}}(\theta, \phi) = r\hat{\mathbf{r}}$$

Line Element:

$$\begin{aligned} d\mathbf{r} &= dr\,\frac{\partial \mathbf{r}}{\partial r} + d\theta\,\frac{\partial \mathbf{r}}{\partial \theta} + d\phi\,\frac{\partial \mathbf{r}}{\partial \phi} \\ &= dr\,\hat{\mathbf{r}} + d\theta\, r\frac{\partial \hat{\mathbf{r}}}{\partial \theta} + d\phi\, r\frac{\partial \hat{\mathbf{r}}}{\partial \phi} \\ &= dr\,\hat{\mathbf{r}} + rd\theta\,\hat{\theta} + r\sin\theta d\phi\,\hat{\phi} \end{aligned}$$

Surface Elements:

Surface	Shape	Normal	Elementary Area
$r = \text{const}$	Sphere	$\hat{\mathbf{r}}$	$r^2 \sin\theta\, d\theta\, d\phi$
$\theta = \text{const}$	Cone	$\hat{\theta}$	$r \sin\theta\, dr\, d\phi$
$\phi = \text{const}$	Half Plane	$\hat{\phi}$	$r\, dr\, d\theta$

Volume Element: $r^2 \sin\theta\, dr\, d\theta\, d\phi$

Spherical Coordinate System

Gradient:

$$\begin{aligned}\nabla f(P) &= \hat{\mathbf{x}}\frac{\partial f}{\partial x} + \hat{\mathbf{y}}\frac{\partial f}{\partial y} + \hat{\mathbf{z}}\frac{\partial f}{\partial z} \\ &= \hat{\mathbf{r}}\frac{\partial f}{\partial r} + \hat{\theta}\frac{1}{r}\frac{\partial f}{\partial \theta} + \hat{\phi}\frac{1}{r\sin\theta}\frac{\partial f}{\partial \phi}\end{aligned}$$

Divergence:

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 F_r\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta F_\theta\right) + \frac{1}{r\sin\theta}\frac{\partial}{\partial \phi}\left(F_\phi\right)$$

Curl:

$$\ldots$$

Laplacian:

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial f}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial \phi^2}\left(F_\phi\right)$$