Physics I

Lecture 9

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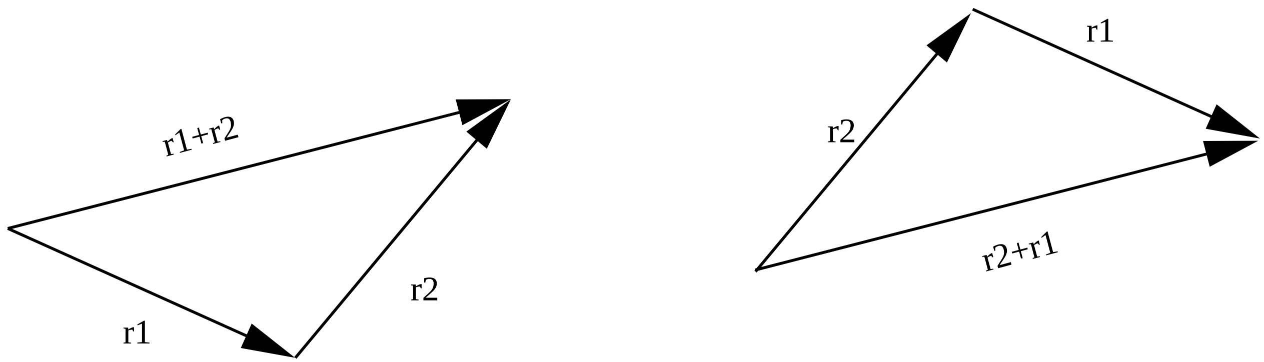
Pure Rotation

- Description of Rotation
- 3 Degrees of freedom
- Generalize idea of rotation about an axis
- Can we write $\theta_x \mathbf{i} + \theta_y \mathbf{j}$ to describe a rotation?

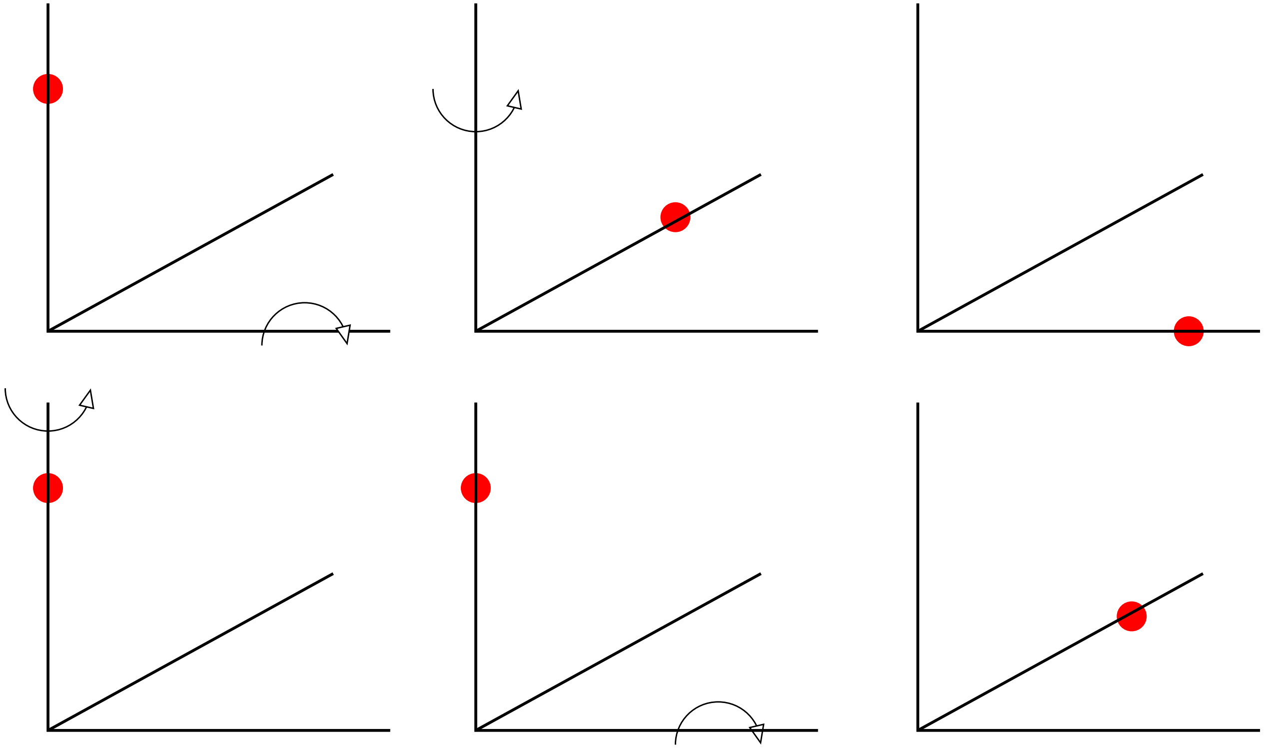
Pure Translation

- Translation has 3 DoF
- Is described by a vector
- Vector addition is commutative



Pure Rotation

Rotation is not a vector. In fact, a matrix is needed to specify a rotation.



Angular Velocity as Vector

- angular velocity vector is defined for fixed axis motion.
- Generalize to instantaneous angular velocity vector.
- Define: $\vec{\omega}$ such that the instantaneous velocity $\vec{v}_i$ of each particle can be written as

$$\vec{v}_i = \vec{\omega} \times \vec{r}_i$$

- Does it exist?
- If we know $\vec{\omega}(t)$, the motion can be found.

Instantaneous Angular Momentum

At any instant t, there is $\vec{\omega}$, such that

$$\begin{aligned}
\mathbf{L} &= \sum_i \mathbf{r}_i \times P_i \\
&= \sum_i m_i \mathbf{r}_i \times (\vec{\omega} \times \mathbf{r}_i) \\
&= \sum_i m_i r_i^2 \vec{\omega} - \sum_i m_i (\mathbf{r}_i \cdot \vec{\omega}) \mathbf{r}_i
\end{aligned}$$

If

$$\begin{aligned}
\vec{\omega} &= \omega_x \mathbf{i} + \omega_y \mathbf{j} + \omega_z \mathbf{k} \\
\mathbf{r}_i &= x_i \mathbf{i} + y_i \mathbf{j} + z_i \mathbf{k}
\end{aligned}$$

Instantaneous Angular Momentum

$$\mathbf{L} = \sum_i m_i r_i^2 \vec{\omega} - \sum_i m_i (\mathbf{r}_i \cdot \vec{\omega}) \mathbf{r}_i$$

The x component of Angular Momentum

$$\begin{aligned}
\mathbf{L}_x &= \sum_i m_i r_i^2 \omega_x - \sum_i m_i (\omega_x x_i + \omega_y y_i + \omega_z z_i) x_i \\
&= \left(\sum m_i (r_i^2 - x_i^2) \right) \omega_x + \left(- \sum m_i y_i x_i \right) \omega_y \\
&\quad + \left(- \sum m_i x_i x_i \right) \omega_z \\
&= I_{xx} \omega_x + I_{xy} \omega_y + I_{xz} \omega_z
\end{aligned}$$

Instantaneous Angular Momentum

- $I_{xx} = (\sum m_i(r_i^2 - x_i^2))$ and I_{yy} and I_{zz} are called Moments of Inertia.
- $I_{xy} = (-\sum m_i y_i x_i)$, I_{xz} and I_{yz} are called Products of Inertia.
- Instantaneous Angular Momentum

(1)
$$\begin{pmatrix} L_x \\ L_y \\ L_z \end{pmatrix} = \begin{pmatrix} I_{xx} & I_{xy} & I_{xz} \\ I_{yx} & I_{yy} & I_{yz} \\ I_{zx} & I_{zy} & I_{zz} \end{pmatrix} \begin{pmatrix} \omega_x \\ \omega_y \\ \omega_z \end{pmatrix}$$

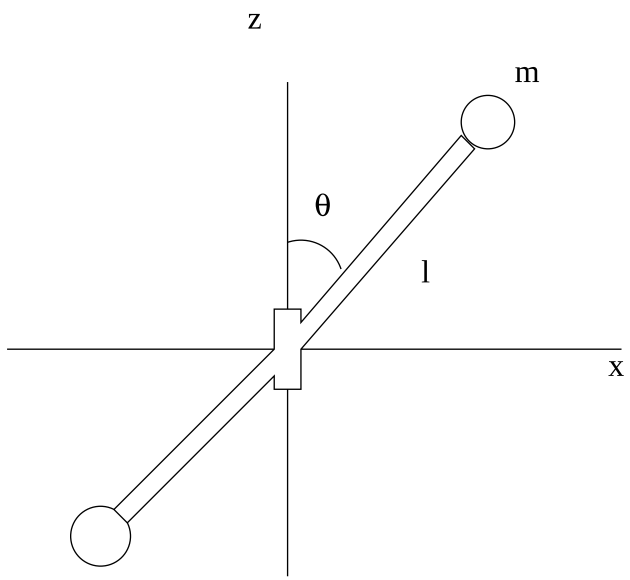
or

(2)
$$\mathbf{L} = I\vec{\omega}$$

Example

Moment of Inertia

$$I = \begin{pmatrix} 2ml^2\cos^2\theta & 0 & -2ml^2\cos\theta\sin\theta \\ 0 & 2ml^2 & 0 \\ -2ml^2\cos\theta\sin\theta & 0 & 2ml^2\sin^2\theta \end{pmatrix}$$



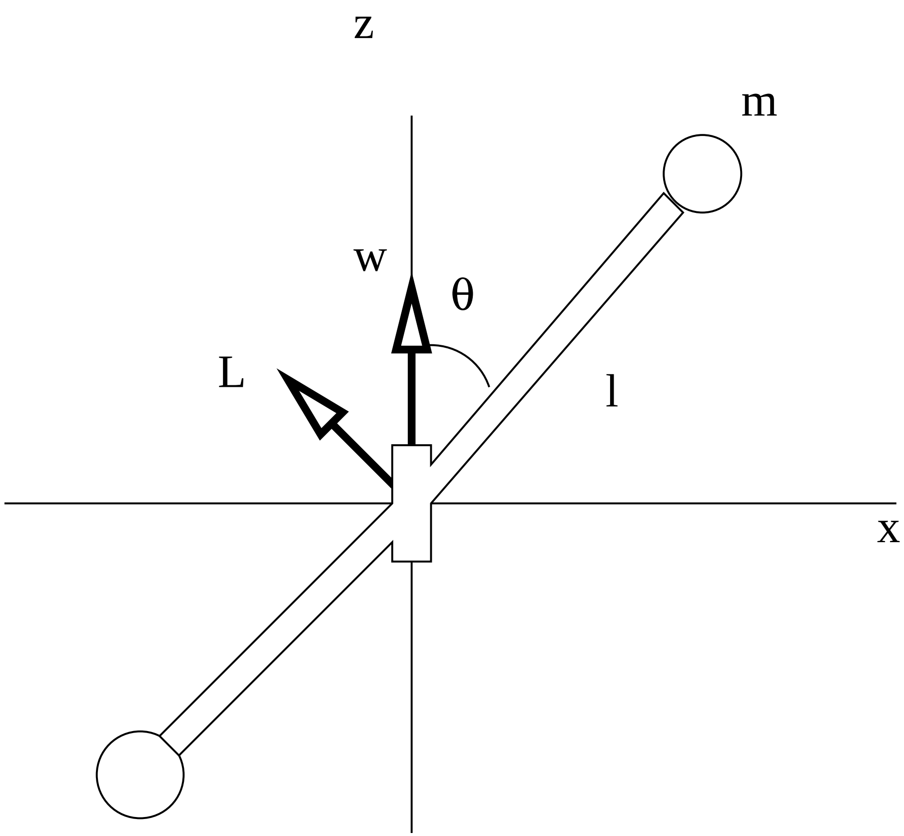
Example

If the body is spinning about z axis $\vec{\omega} = \omega_z \mathbf{k}$ then,

$$L = \begin{pmatrix} 2ml^2\cos^2\theta & 0 & -2ml^2\cos\theta\sin\theta \\ 0 & 2ml^2 & 0 \\ -2ml^2\cos\theta\sin\theta & 0 & 2ml^2\sin^2\theta \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ \omega_z \end{pmatrix}$$

$$= \begin{pmatrix} -2ml^2\cos\theta\sin\theta\omega_z \\ 0 \\ 2ml^2\sin^2\theta\omega_z \end{pmatrix}$$

Example



Example

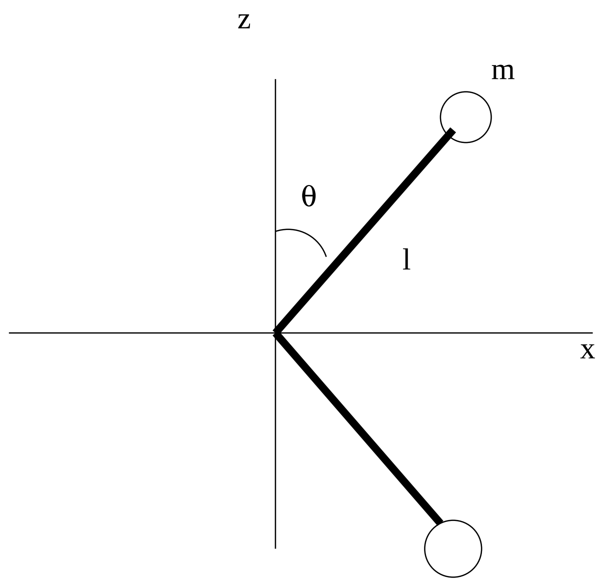
By the time body turns and lies in YZ plane, Moment of Inertia becomes

$$I = \begin{pmatrix} 2ml^2 & 0 & 0 \\ 0 & 2ml^2\cos^2\theta & -2ml^2\cos\theta\sin\theta \\ 0 & -2ml^2\cos\theta\sin\theta & 2ml^2\sin^2\theta \end{pmatrix}$$

Principle Axes

Moment of Inertia

$$I = \begin{pmatrix} 2ml^2\cos^2\theta & 0 & 0 \\ 0 & 2ml^2 & 0 \\ 0 & 0 & 2ml^2\sin^2\theta \end{pmatrix}$$



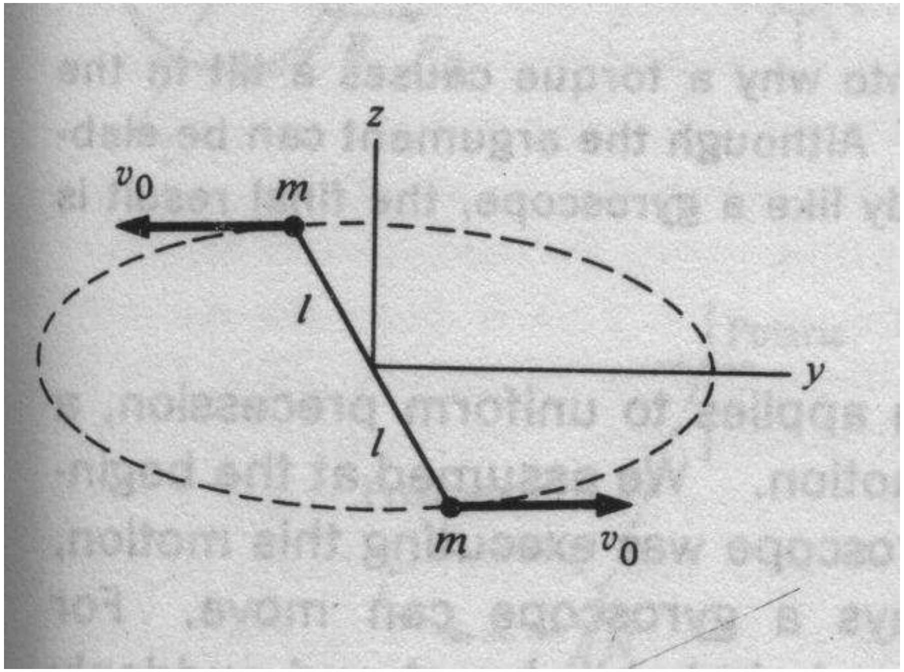
The three axes are called the Principle Axes of the body.

Moment of Inertia: Properties

- Moment of Inertia is a geometric quantity
- As body moves in space, MI changes
- For every body, Principle Axes exist.
- In body fixed coordinate system, MI remains constant!

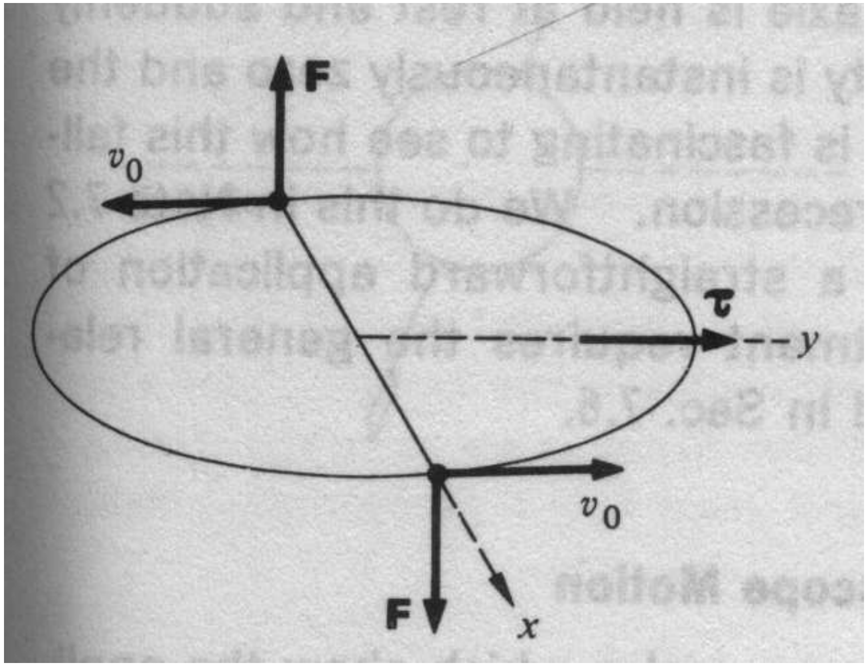
Dynamics

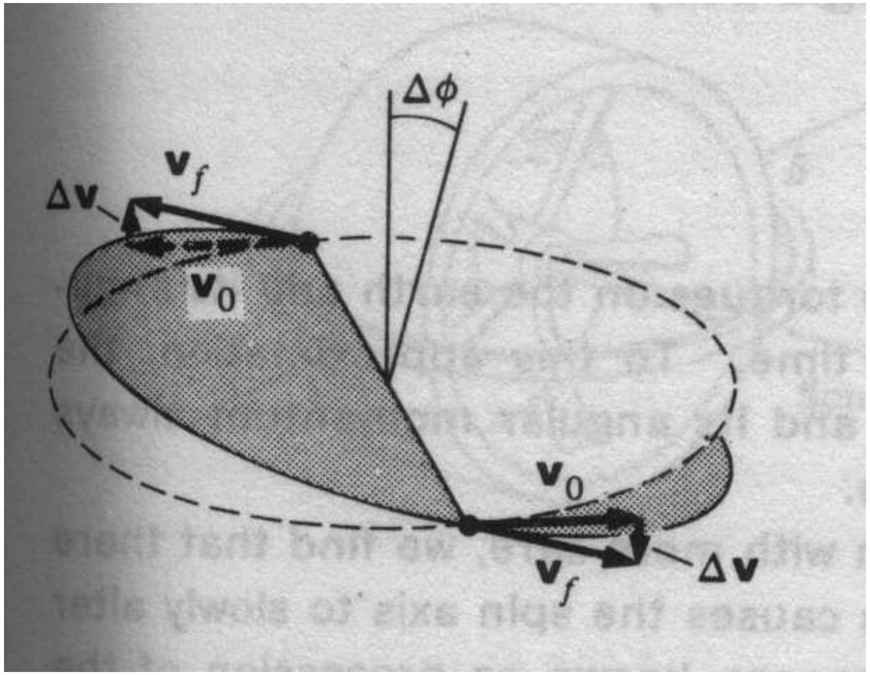
If we apply torque to a steady rod in y direction, the plane of the rod spins about y axis.



Dynamics

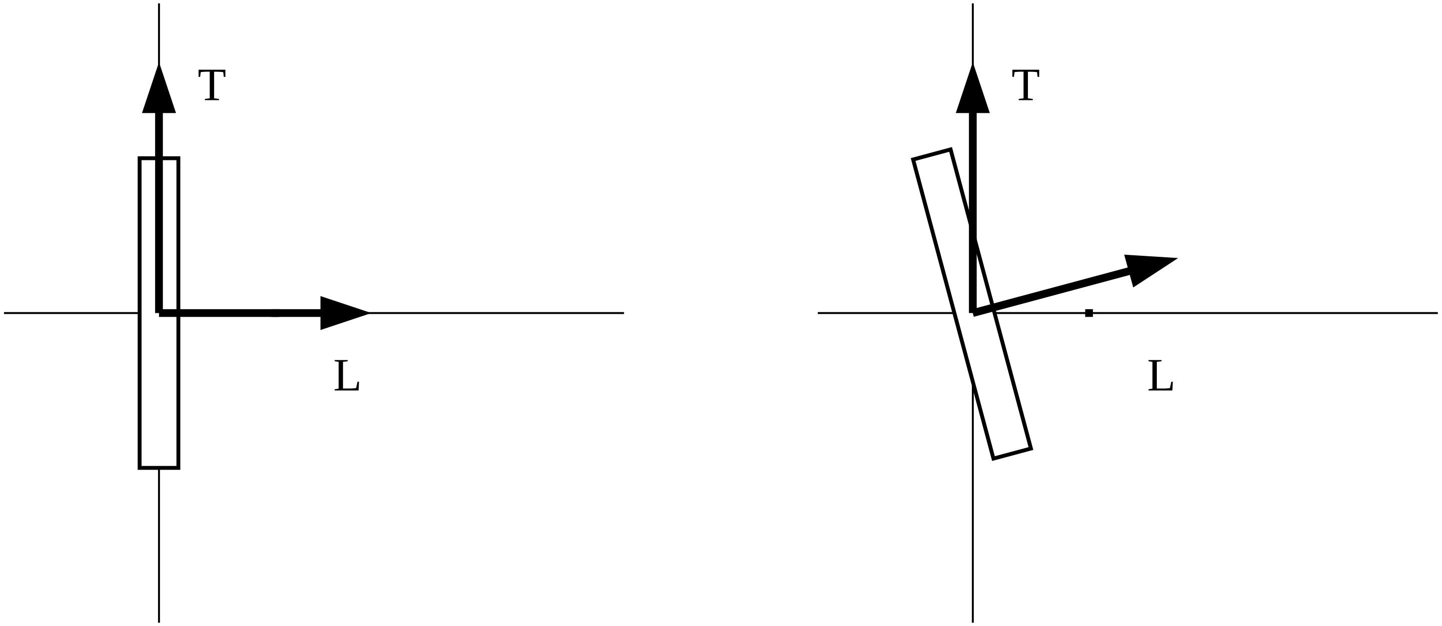
If the rod is already spinning about z axis. Now torque is applied in y direction, the plane of rotation of the rod spins about x axis! The change in angular momentum is in the direction of torque, which is expected.





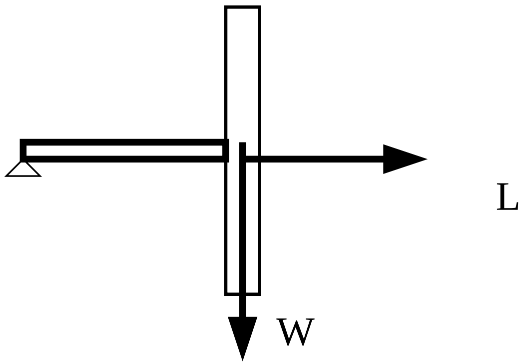
Dynamics

Motion of a spinning disc. Angular momentum changes its direction so as to align itself with the direction of the torque.

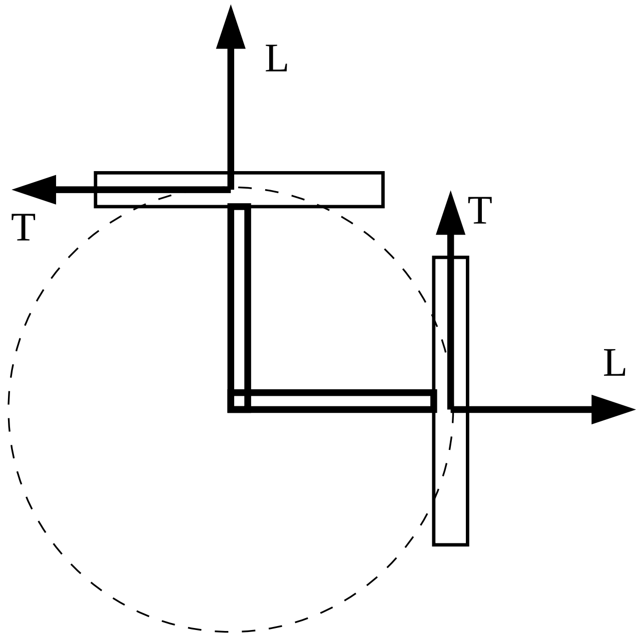


Gyroscope

Now to consider the cycle wheel shown in the video.



Side View



Top View

Gyroscope

Let us assume that the gyroscope is spinning about vertical axis with angular speed Ω. The torque is lW. Then

$$\begin{aligned} \frac{dL}{dt} &= \Omega L \\ &= lW \\ \Omega &= \frac{lW}{L} \end{aligned}$$