Physics I

Lecture 7

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Angular Momentum

- Angular Momentum of a particle is defined as

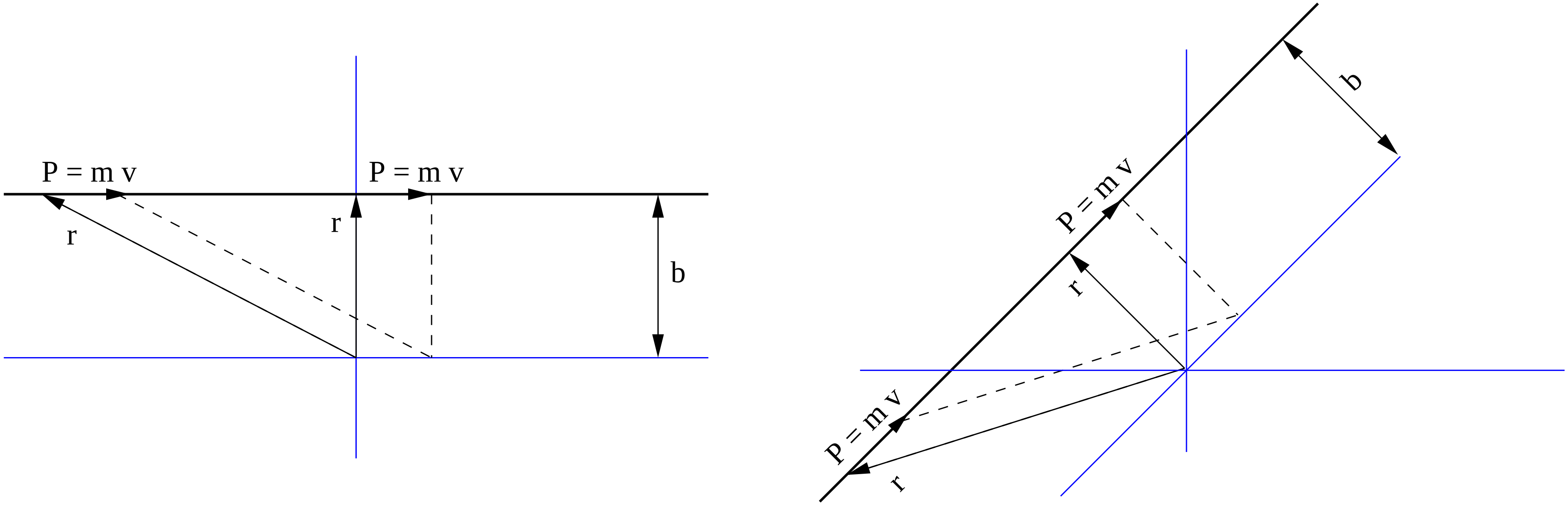
$$\mathbf{L} = \mathbf{r} \times \mathbf{P}$$

- The magnitude of $\mathbf{L}$ is

$$|\mathbf{L}| = rP\sin(\phi)$$

  and is equal to the area of the parallelogram enclosed by $\mathbf{r}$ and $\mathbf{P}$.
- The direction of $\mathbf{L}$ is perpendicular to the plane of $\mathbf{r}$ and $\mathbf{P}$.

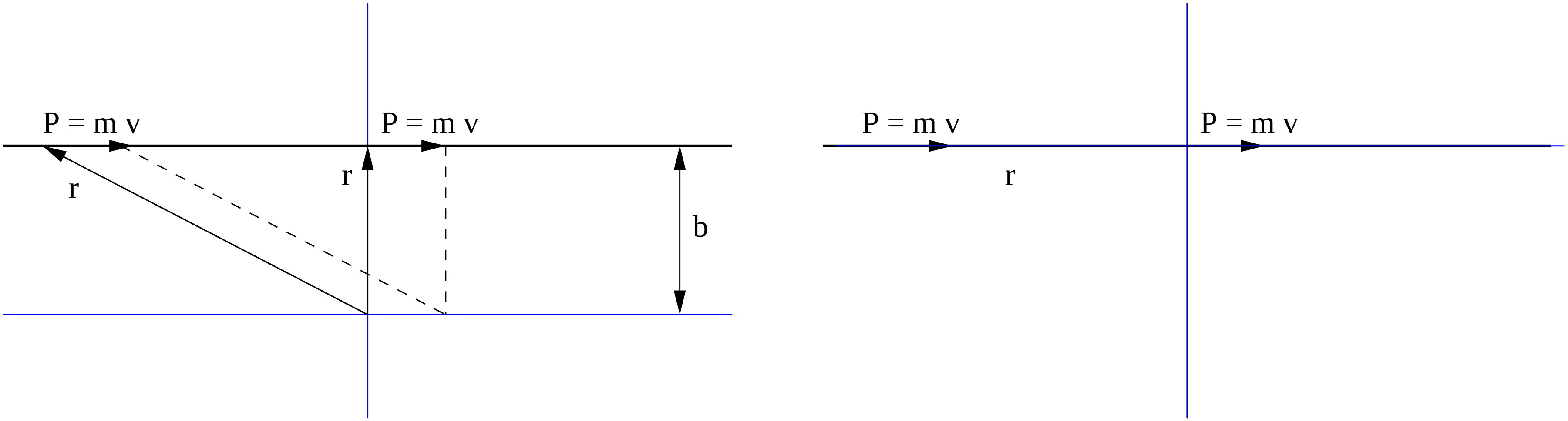
Free Particle



Angular momentum in both cases is $mvb\mathbf{k}$ and is a constant of motion.

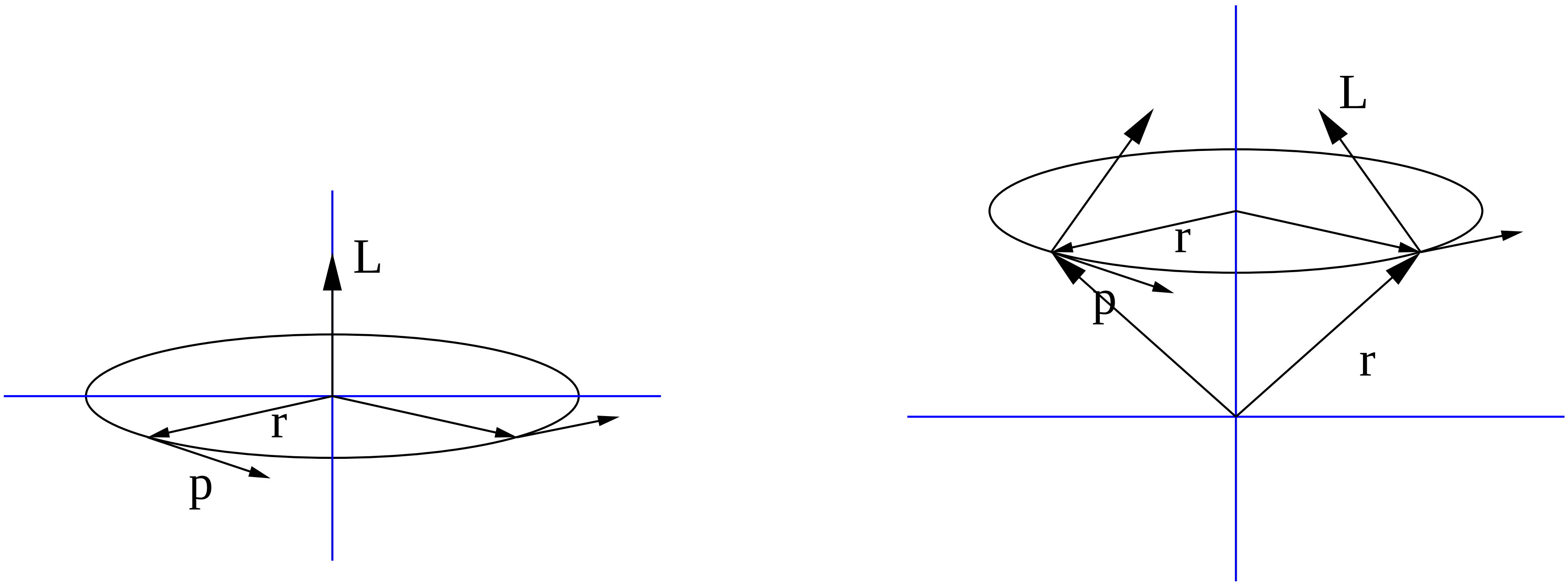
Choice of orientation of coordinate system is not important.

Free Particle



Angular momentum in the second case is 0! Choice of origin is of the coordinate system is important!

Circular Trajectory



(a) Angular momentum $= mvr\mathbf{k}$ and is constant of motion. (b) Angular momentum $= mvr\mathbf{k} + mvz\hat{\mathbf{r}}$ and is not constant.(cylindrical coordinates)

Torque

Let $\mathbf{F}$ be a force on a particle when it is at $\mathbf{r}$. The torque

$$\tau = \mathbf{r} \times \mathbf{F}$$

Just like angular momentum, the torque depends on the choice of the origin.

Newton's Second Law?

$$\begin{aligned}
\frac{d\mathbf{L}}{dt} &= \frac{d}{dt}\left(\mathbf{r} \times \mathbf{P}\right) \\
&= \left(\frac{d}{dt}\mathbf{r}\right) \times \mathbf{P} + \mathbf{r} \times \left(\frac{d}{dt}\mathbf{P}\right) \\
&= \mathbf{r} \times \mathbf{F} \\
&= \tau
\end{aligned}$$

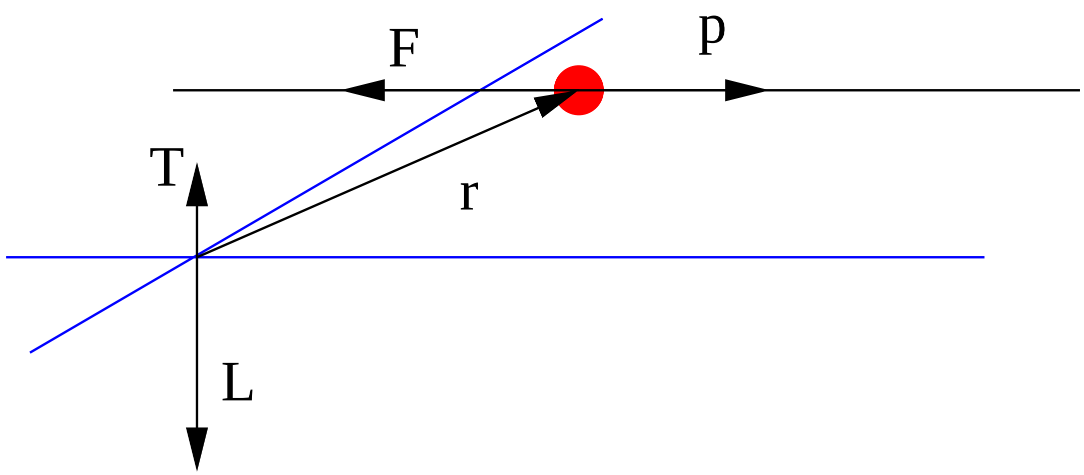
Rectilinear Motion

Initial Position $(0, b, 0)$

Initial velocity $u_0\mathbf{i}$

Frictional force $-\mu m g\mathbf{i}$

$\mathbf{L} \| \tau$



$$
\begin{aligned}
L &= L_0 - \tau t \\
mbu(t) &= mbu_0 - \mu mgbt \\
u(t) &= u_0 - \mu gt
\end{aligned}
$$

Systems of Particles

Total angular momentum $\mathbf{L}$

$$\mathbf{L} = \sum_i \mathbf{l}_i$$

Then,

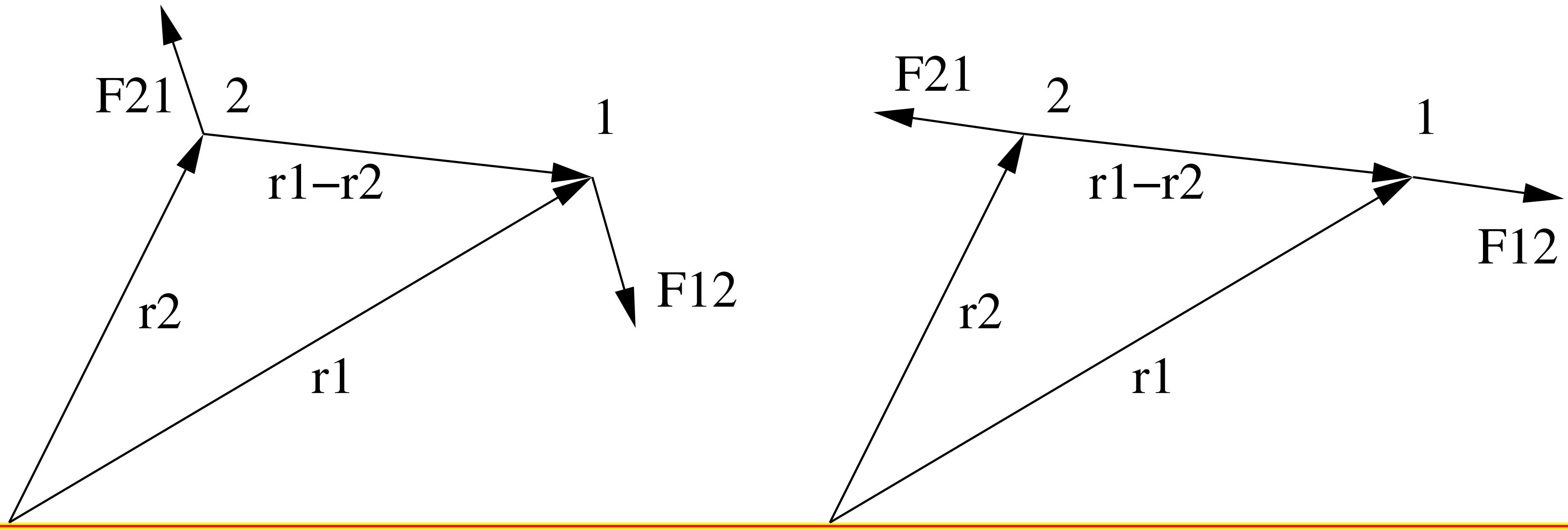
$$\begin{aligned} \frac{d\mathbf{L}}{dt} &= \sum_i \frac{d\mathbf{l}_i}{dt} \\ &= \sum_i \tau_i \\ &= \sum_i \mathbf{r}_i \times \left(F_i^{int} + F_i^{ext}\right) \end{aligned}$$

Systems of Particles

If $\sum_i \mathbf{r}_i \times F_i^{int} = 0$, then

$$\frac{d\mathbf{L}}{dt} = \sum_i \mathbf{r}_i \times F_i^{ext}$$

$$\frac{d\mathbf{L}}{dt} = \sum_i \tau_i^{ext}$$



Rigid Bodies

- Many particle systems
- Inter-particle distance is fixed.

$$|\mathbf{r}_i - \mathbf{r}_j| = C_{ij}$$

- Retains shape and size
- 6 degrees of freedom

Fixed Axis Rotation

- All particles rotate about an axis fixed in space.
- All particles have same instantaneous angular speed ω
- Choose z-axis along the axis of rotation, $\vec{\omega} = \omega\mathbf{k}$

Fixed Axis Rotation

$$
\begin{aligned}
\mathbf{L} &= \sum_i \mathbf{r}_i \times P_i \\
&= \sum_i m_i \mathbf{r}_i \times (\vec{\omega} \times \mathbf{r}_i) \\
&= \sum_i m_i r_i^2 \vec{\omega} - \sum_i m_i (\mathbf{r}_i \cdot \vec{\omega}) \mathbf{r}_i
\end{aligned}
$$

Hence,

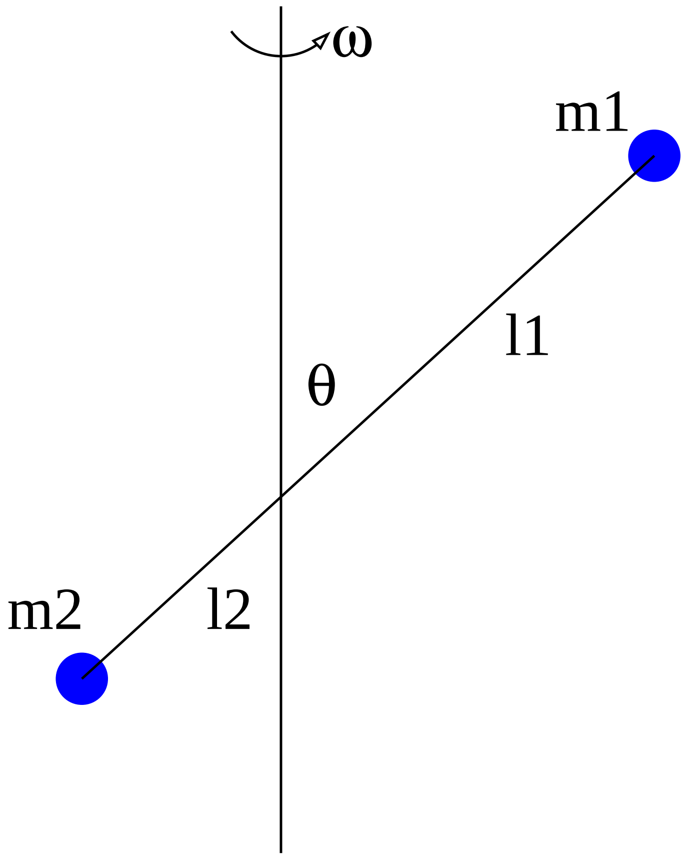
$$
\begin{aligned}
\mathbf{L}_z &= \sum_i m_i \left(r_i^2 - z_i^2 \right) \omega \hat{\mathbf{k}} \\
&= \sum_i m_i \left(x_i^2 + y_i^2 \right) \omega \hat{\mathbf{k}} \\
&= I_{zz} \vec{\omega}
\end{aligned}
$$

Moment of Inertia

- I_{zz} is called moment of inertia.
- purely geometrical quantity. Does not depend on motion.
- For continuous bodies

$$
\begin{aligned}
I_{zz} &= \int (x^2 + y^2) dm \\
&= \int (x^2 + y^2) \rho dv
\end{aligned}
$$

Examples



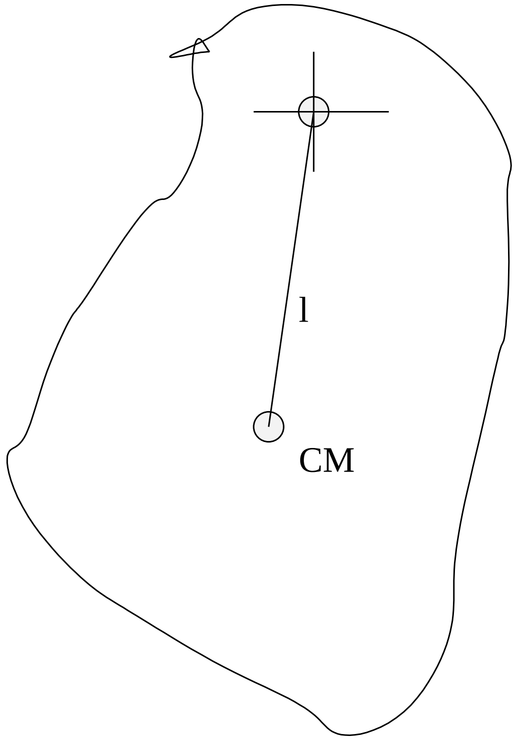
Angular Momentum

$$L_z = m(l_1^2 + l_2^2)\omega \sin^2\theta$$

Parallel Axis Theorem

Let I be MI of a body about an axis. The distance of CM from the axis be l. If I_0 is MI of the body about an axis that is parallel to the original axis but passing through CM of the body, then

$$I = I_0 + Ml^2$$



Dynamics of Rotation

In case of fixed axis, $L_z = I_{zz}\omega$, hence

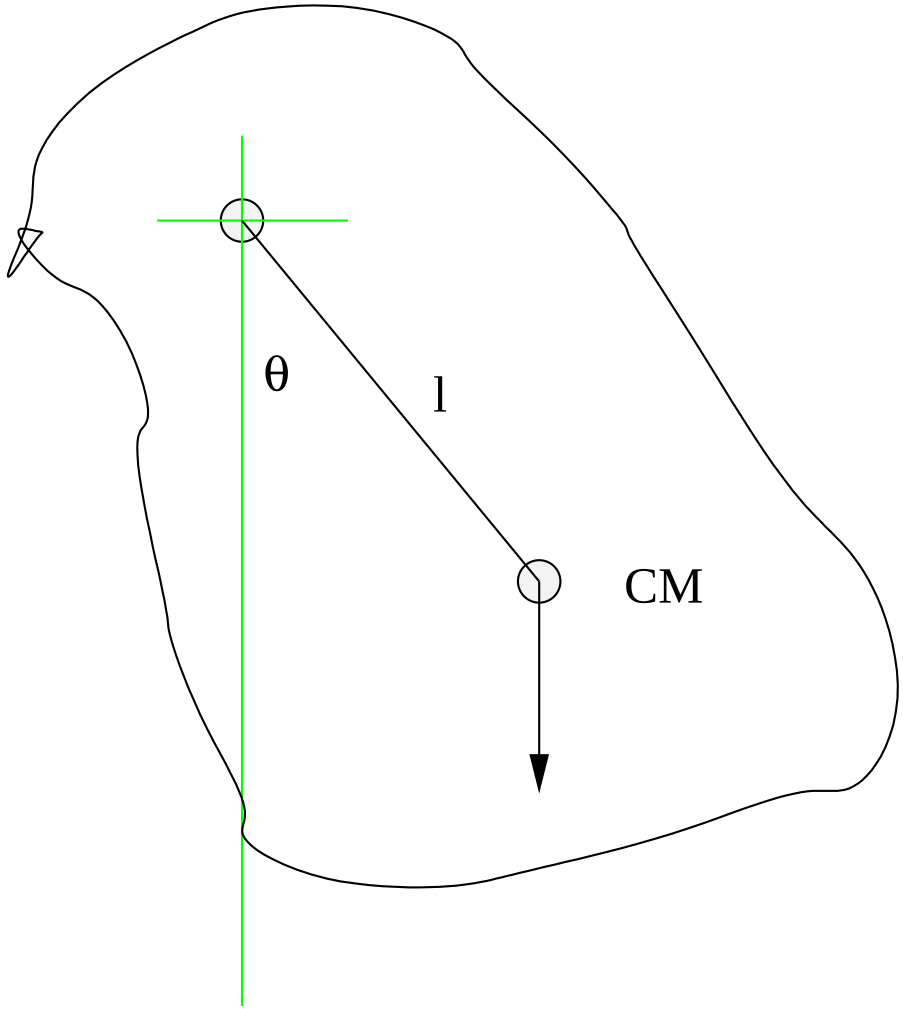
$$\begin{aligned} \tau_z &= \frac{dL_z}{dt} \\ &= I\frac{d\omega}{dt} \\ &= I\alpha \end{aligned}$$

The kinetic energy of such rotation is

$$\begin{aligned} K &= \frac{1}{2}\sum m_i v_i^2 \\ &= \frac{1}{2}Iw^2 \end{aligned}$$

Physical Pendulum

$$\begin{aligned}\tau &= \int \vec{r} \times \vec{g} dm \\ &= \left(\int \vec{r} dm \right) \times \vec{g} \\ &= \vec{R}_{cm} \times M\vec{g} \\ &= \vec{R}_{cm} \times \vec{W}\end{aligned}$$



Physical Pendulum

$$
\begin{aligned}
I\frac{d\omega}{dt} &= \vec{R}_{cm} \times \vec{W} \\
I\ddot{\theta} &= -mgl\sin\theta \approx -mgl\theta \\
\Rightarrow \theta &= A\sin(\Omega t + \phi)
\end{aligned}
$$

The time period of oscillations

$$
T = \frac{2\pi}{\Omega} = \sqrt{\frac{I}{lmg}} = \sqrt{\frac{k^2 + l^2}{gl}}
$$

Physical Pendulum