Physics I

Lecture 6

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Energy Conservation

A particle moves from position A to position B under influence of a conservative force $\mathbf{F}$. From work-energy theorem,

$$K_B - K_A = \int_A^B \mathbf{F} \cdot dr$$

And from the definition of potential energy

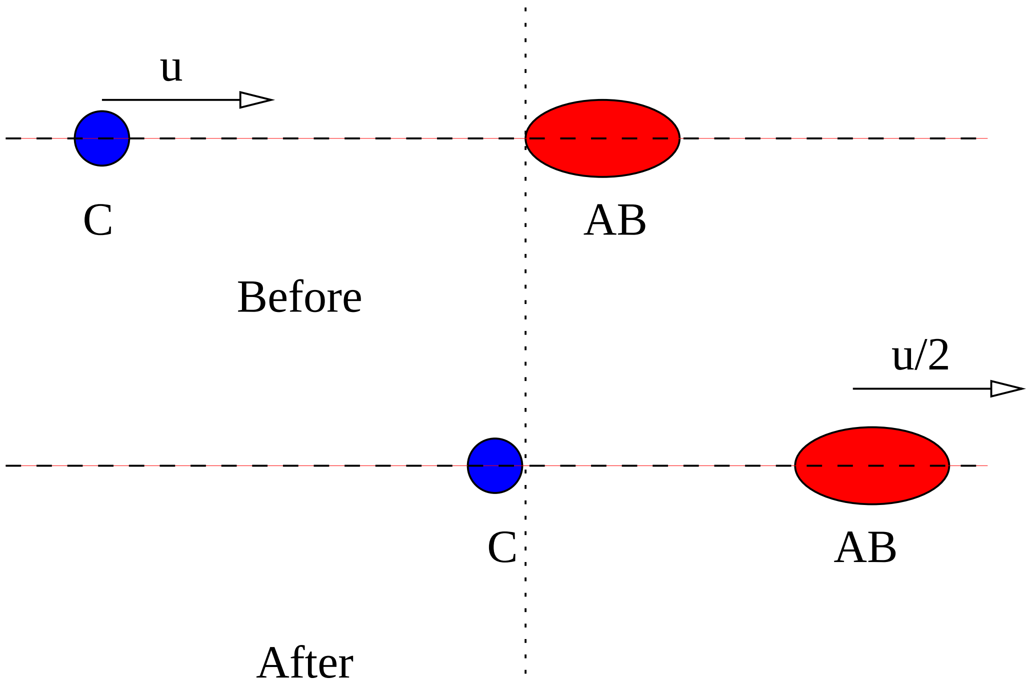
$$\int_A^B \mathbf{F} \cdot dr = V_A - V_B$$

For any pair of points

$$K_A + V_A = K_B + V_B$$

The total energy, defined as $K_A + V_A$ remains constant.

Energy Conservation



Suppose a particle, say C, of mass m collides with an object (like a molecule) of mass $2m$. Initially the object is at rest and the speed of C is u_0. After the collision, it is found that C has stopped and the object is moving with speed $u_0/2$.

Energy Conservation

$$
\begin{aligned}
\text{InitialMom} &= mu_0 + 0 = mu_0 \\
\text{FinalMom} &= 0 + (2m)\frac{u_0}{2} = mu_0
\end{aligned}
$$

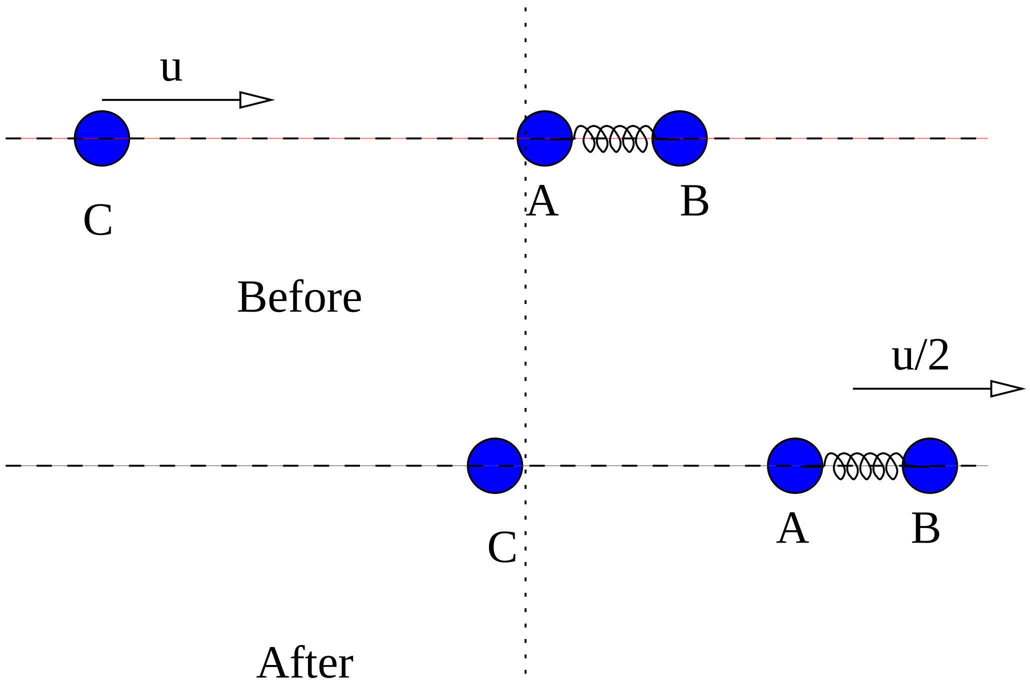
Momentum conservation is satisfied.

$$
\begin{aligned}
\text{InitialK} &= \frac{1}{2}mu_0^2 \\
\text{FinalK} &= \frac{1}{2}(2m)u_0^2 = \frac{1}{4}mu_0^2
\end{aligned}
$$

Energy conservation is not satisfied. Kinetic energy of $1/4mu_0^2$ is lost.

The collision is inelastic.

Energy Conservation



Let us look at the object (molecule) more closely. Suppose the molecule consists of two particles of mass m each and joined by a spring. So the original collision problem can be seen as a simple mechanical problem of three particles and a spring.

Energy Conservation

If the collision has occurred at $t = 0$, the solutions are given by

$$\begin{aligned} x_A(t) &= \frac{u_0}{2}t + \frac{u_0}{2\omega}\sin(\omega t) \\ x_B(t) &= l + \frac{u_0}{2}t - \frac{u_0}{2\omega}\sin(\omega t) \end{aligned}$$

where $\omega = \sqrt{2k/m}$. The potential energy stored in the spring is

$$\begin{aligned} V_{vib} &= \frac{1}{2}k(x_B - x_A - l)^2 \\ &= \frac{1}{4}mu_0^2\sin^2(\omega t) \end{aligned}$$

Energy Conservation

The velocities by

$$\begin{aligned} \dot{x}_A(t) &= \frac{u_0}{2} + \frac{u_0}{2}\cos(\omega t) \\ \dot{x}_B(t) &= \frac{u_0}{2} - \frac{u_0}{2}\cos(\omega t) \end{aligned}$$

The kinetic energy can be written as

$$\begin{aligned} K_A + K_B &= \frac{1}{4}mu_0^2 + \frac{1}{4}mu_0^2\cos^2(\omega t) \\ &= K_{cm} + K_{vib} \end{aligned}$$

where $K_{cm} = \frac{1}{4}mu_0^2$ and $K_{vib} = \frac{1}{4}mu_0^2\cos^2(\omega t)$

Energy Conservation

Energy of the system after the collision is

$$K_C + K_A + K_B + V_{vib} = 0 + K_{cm} + K_{vib} + V_{vib}$$

But before we considered A + B + Spring as one system and ignored the energy of motion that is internal to the system. This energy $K_{vib} + V_{vib}$ will be called as internal energy of the system.

Energy Conservation

1. Fundamental forces are believed to be conservative.
2. Internal Energies manifest themselves in many forms: Mechanical Energy, Heat, Radiation Energy, Nuclear Energy etc.

Total Energy of an isolated system is conserved

Potential Energy and Equilibrium

Since, for a particle, $\mathbf{F} = -\nabla U$, where U is potential energy, the condition for equilibrium is given by

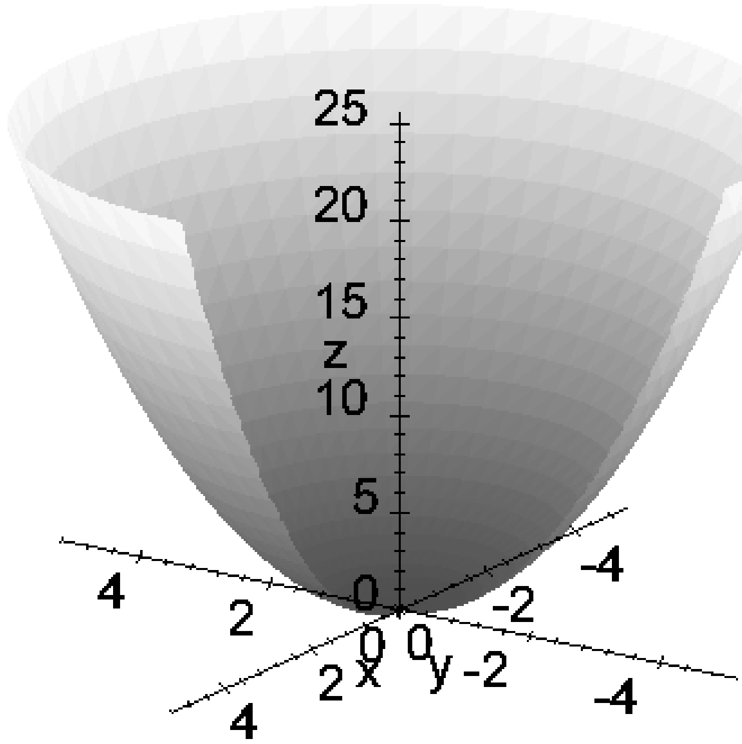
$$\nabla U = 0$$

This means

$$\begin{aligned} \frac{\partial U}{\partial x} &= 0 \\ \frac{\partial U}{\partial y} &= 0 \end{aligned}$$

In one dimension, the three types of equilibrium are possible.

Examples



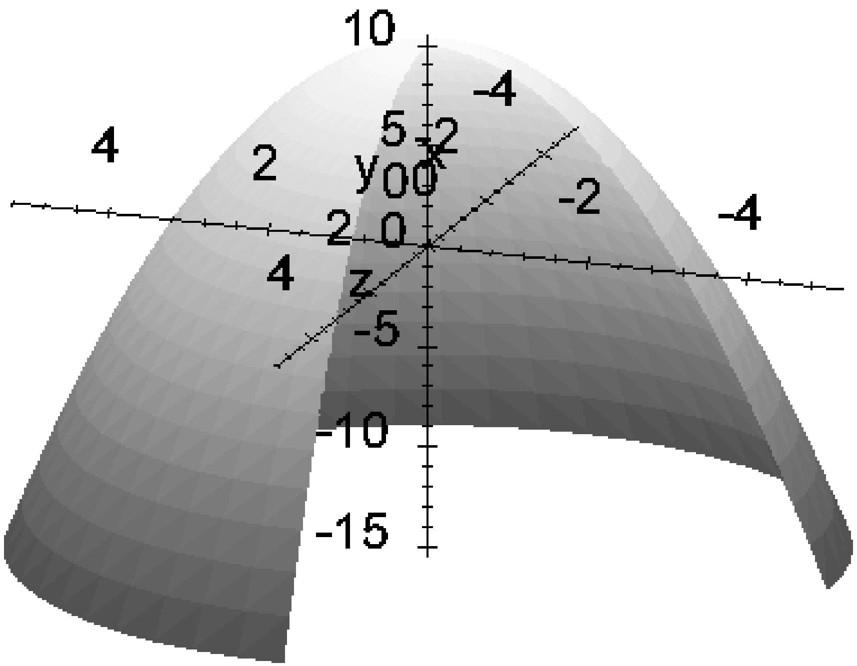
$$U(x,y) = x^2 + y^2$$

Equilibrium at $(0,0)$

Jacobian

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial^2 U}{\partial^2 x} & \frac{\partial^2 U}{\partial y \partial x} \\ \frac{\partial^2 U}{\partial y \partial x} & \frac{\partial^2 U}{\partial^2 y} \end{vmatrix} \\ &= \begin{vmatrix} 2 & 0 \\ 0 & 2 \end{vmatrix} \\ &= 4 \end{aligned}$$

Examples



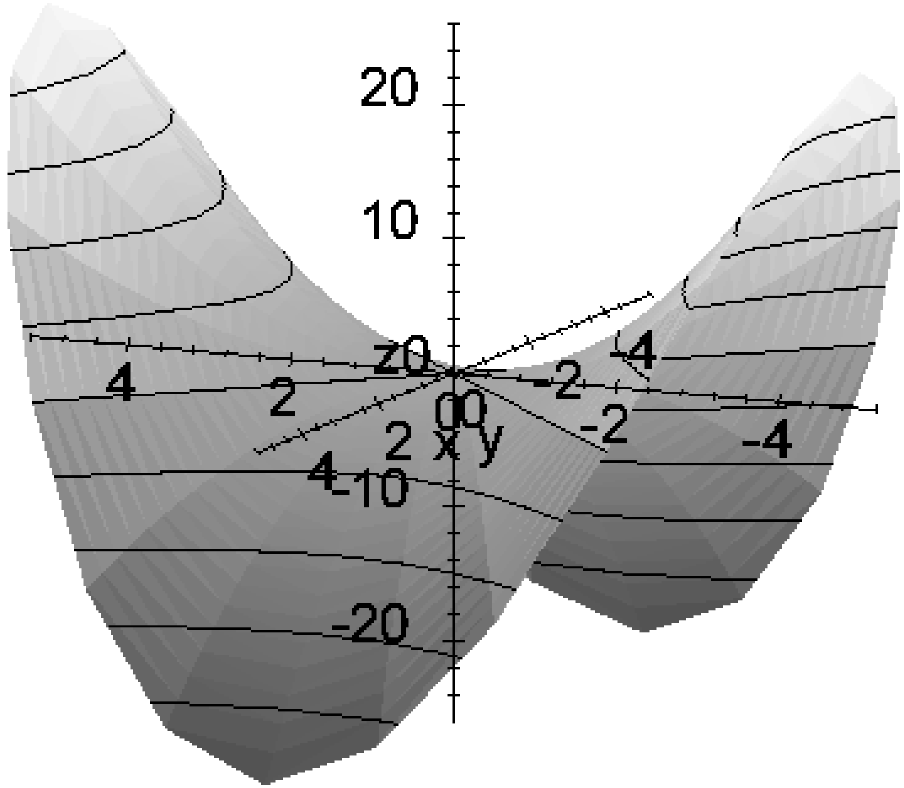
$$U(x, y) = -x^2 - y^2$$

Equilibrium at $(0, 0)$

Jacobian

$$\begin{aligned} J &= \begin{vmatrix} \frac{\partial^2 U}{\partial^2 x} & \frac{\partial^2 U}{\partial y \partial x} \\ \frac{\partial^2 U}{\partial y \partial x} & \frac{\partial^2 U}{\partial^2 y} \end{vmatrix} \\ &= \begin{vmatrix} -2 & 0 \\ 0 & -2 \end{vmatrix} \\ &= 4 \end{aligned}$$

Examples



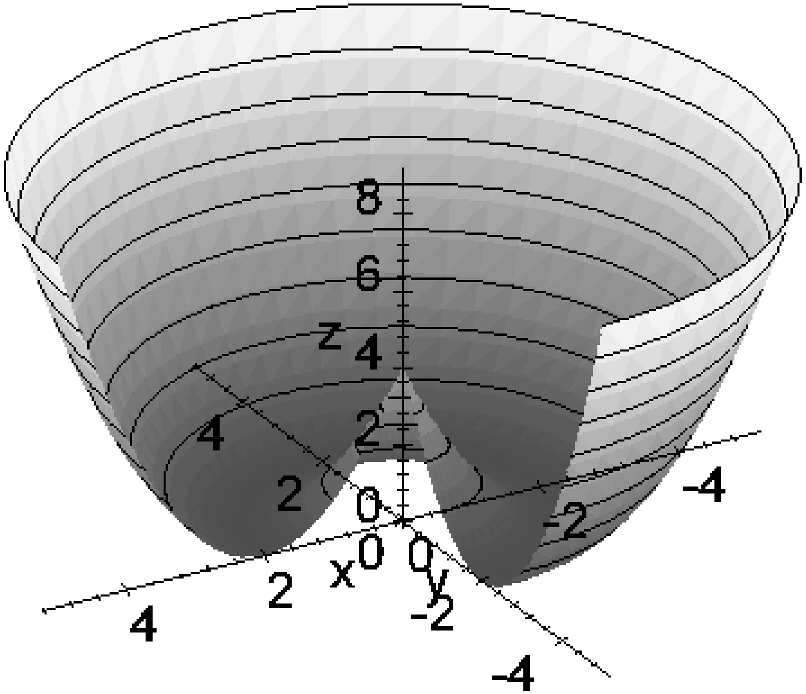
$$U(x, y) = x^2 - y^2$$

Equilibrium at $(0, 0)$

Jacobian

$$J = \begin{vmatrix} \frac{\partial^2 U}{\partial^2 x} & \frac{\partial^2 U}{\partial y \partial x} \\ \frac{\partial^2 U}{\partial y \partial x} & \frac{\partial^2 U}{\partial^2 y} \end{vmatrix} = \begin{vmatrix} 2 & 0 \\ 0 & -2 \end{vmatrix} = -4$$

Examples



$$U(x,y) = (r-2)^2$$

Equilibrium: Circle $r = 2$

Jacobian

$$J = \begin{vmatrix} \frac{\partial^2 U}{\partial^2 x} & \frac{\partial^2 U}{\partial y \partial x} \\ \frac{\partial^2 U}{\partial y \partial x} & \frac{\partial^2 U}{\partial^2 y} \end{vmatrix}$$

$$= \begin{vmatrix} 2 & 0 \\ 0 & 0 \end{vmatrix}$$

$$= 0$$