Physics I

Lecture 5

Charudatt Kadolkar

IIT Guwahati

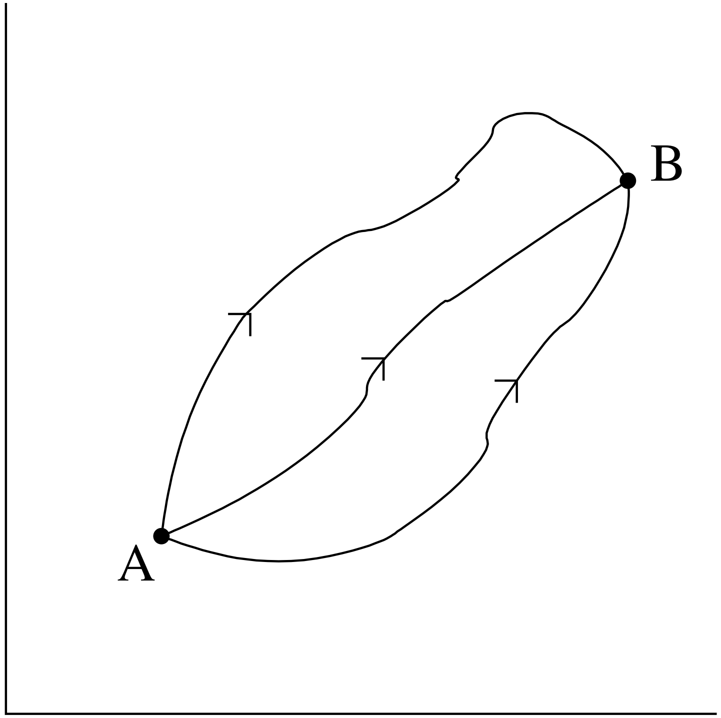
Conservative Force

- A Force Field $\mathbf{F}$ is conservative if work done by the force between **any** two points in independent of the path.

That is the work

$$W_{AB} = \int_A^B \mathbf{F} \cdot d\mathbf{r}$$

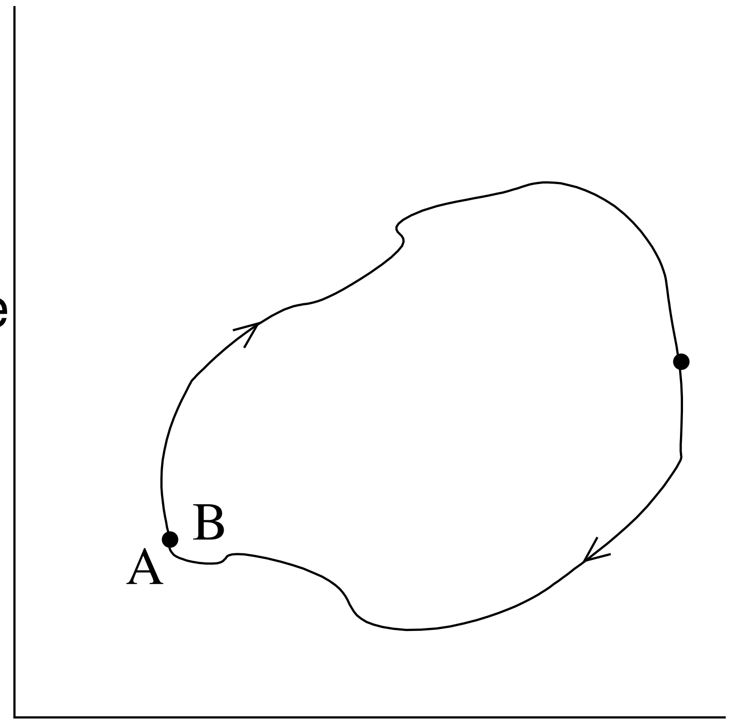
depends only on end-points A and B.



Conservative Force

- A path is called a closed path (loop) when the end-point is same as starting-point.

A force is conservative if and only if work done along **every** closed path is zero.

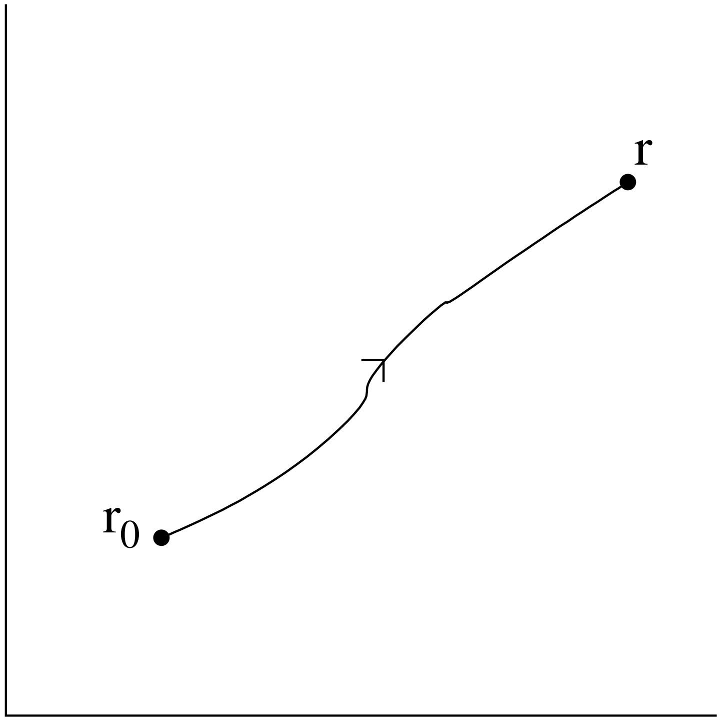


Potential Energy

- Let $\mathbf{F}$ be conservative force.
- Choose an arbitraty point $\mathbf{r}_0$
- Define a scalar function

$$U(\mathbf{r}) = -\int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}'$$

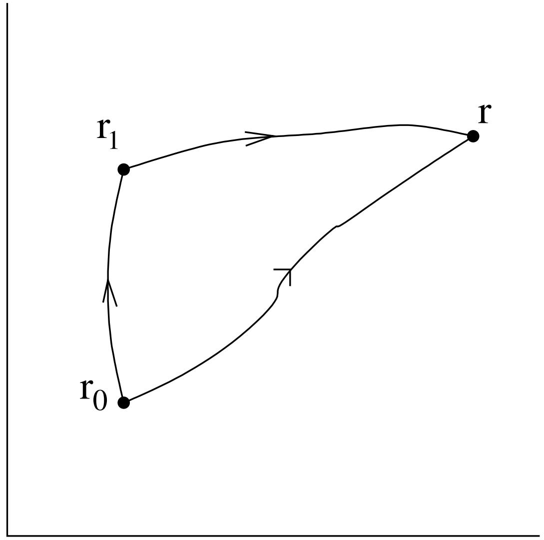
- The function U is called potential energy function.
- Is well defined since the integral is path independent.



Potential Energy

- If we choose r_1 instead of r_0 as a reference point, we get different potential energy function, say V.

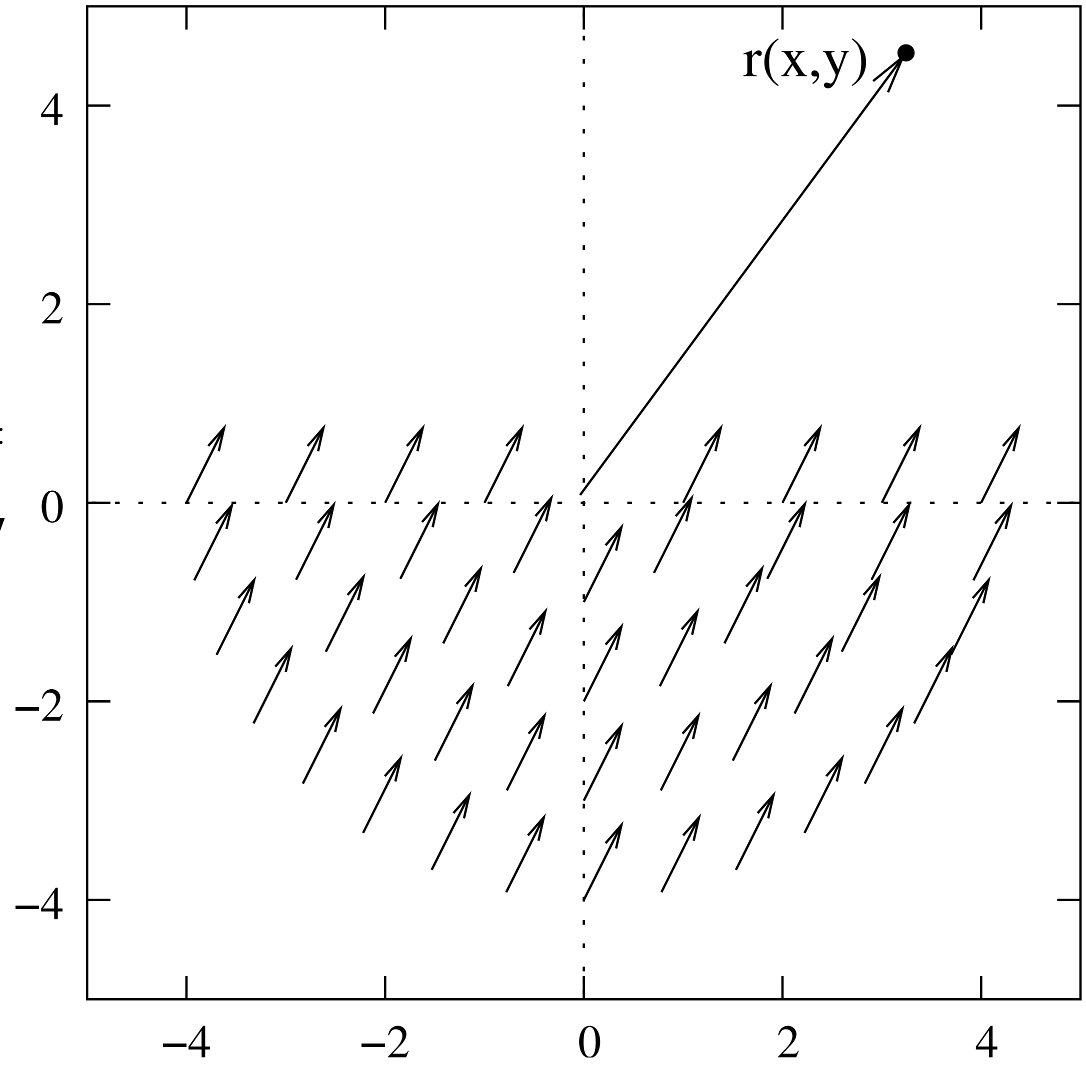
$$
\begin{aligned}
V(\mathbf{r}) &= -\int_{\mathbf{r}_1}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' \\
&= -\int_{\mathbf{r}_1}^{\mathbf{r}_0} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' - \int_{\mathbf{r}_0}^{\mathbf{r}} \mathbf{F}(\mathbf{r}') \cdot d\mathbf{r}' \\
&= U(\mathbf{r}) - U(\mathbf{r}_1)
\end{aligned}
$$



The two functions differ only by a constant.

Example

Let $\mathbf{F} = \mathbf{i} + 2\mathbf{j}$. Choose $\mathbf{r}_0 = (0, 0)$. Let $\mathbf{r} = (x, y)$ be any arbitrary point.



Example

Choose a straight line joining $\mathbf{r}_0$ to $\mathbf{r}$. Then $d\mathbf{r} = \mathbf{i}dx + \mathbf{j}dy$.

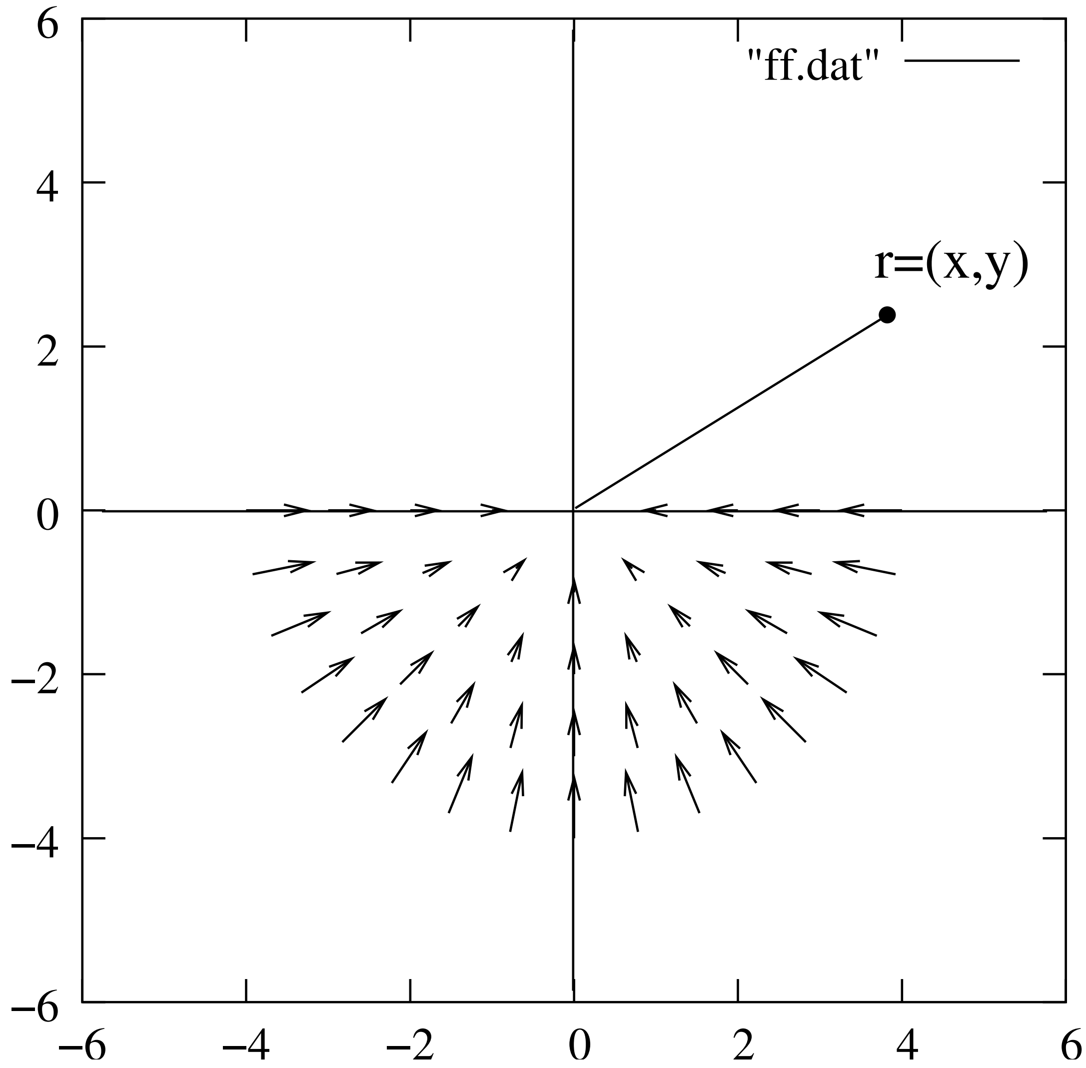
$$\begin{aligned} U(\mathbf{r}) &= -\int \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \\ &= -\int_0^x F_x dx - \int_0^y F_y dy \\ &= -x - 2y \end{aligned}$$

Example

Let $\mathbf{F} = -kx\mathbf{i} - ky\mathbf{j}$

Choose $\mathbf{r}_0 = (0, 0)$.

Let $\mathbf{r} = (x, y)$ be any arbitrary point.

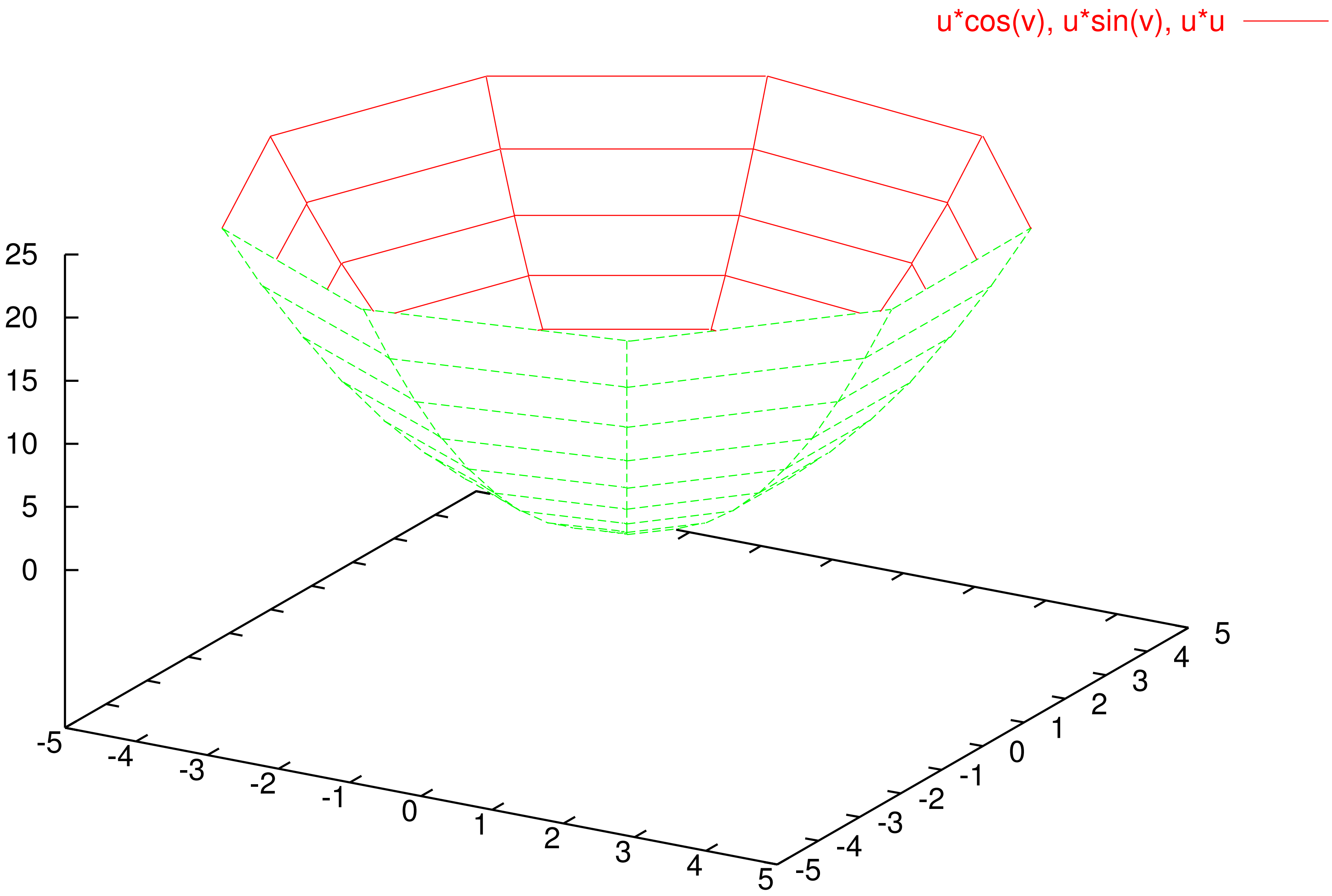


Example

Choose a straight line joining $\mathbf{r}_0$ to $\mathbf{r}$. Then $d\mathbf{r} = \mathbf{i}dx + \mathbf{j}dy$.

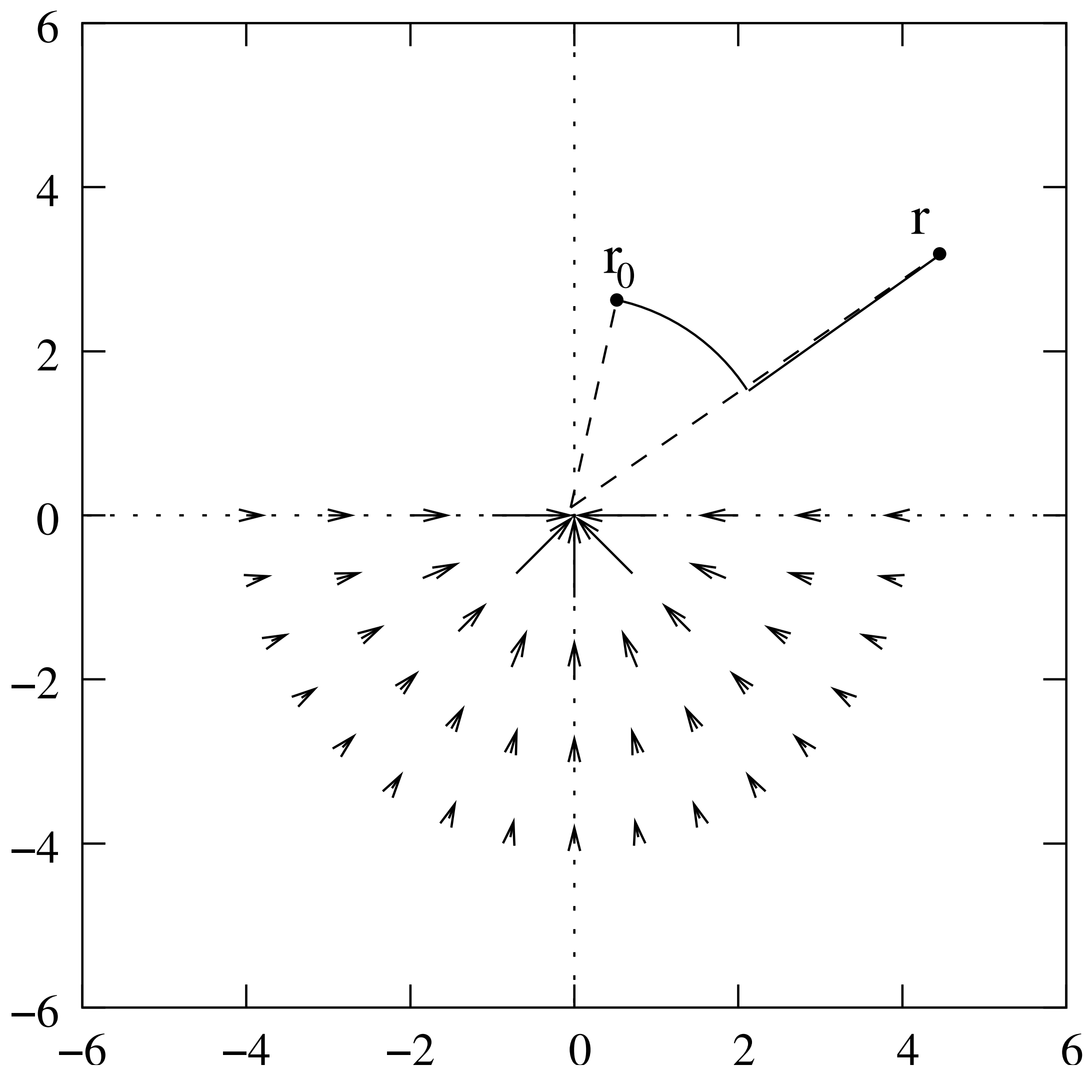
$$
\begin{aligned}
U(\mathbf{r}) &= -\int \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \\
&= -\int_0^x F_x dx - \int_0^y F_y dy \\
&= k\frac{x^2}{2} + k\frac{y^2}{2} = \frac{1}{2}kr^2
\end{aligned}
$$

Example



Example

Let $\mathbf{F} = -k\frac{\hat{\mathbf{r}}}{r^2}$. Choose $\mathbf{r}_0 = (x_0, y_0)$. Let $\mathbf{r} = (x, y)$ be any arbitrary point.



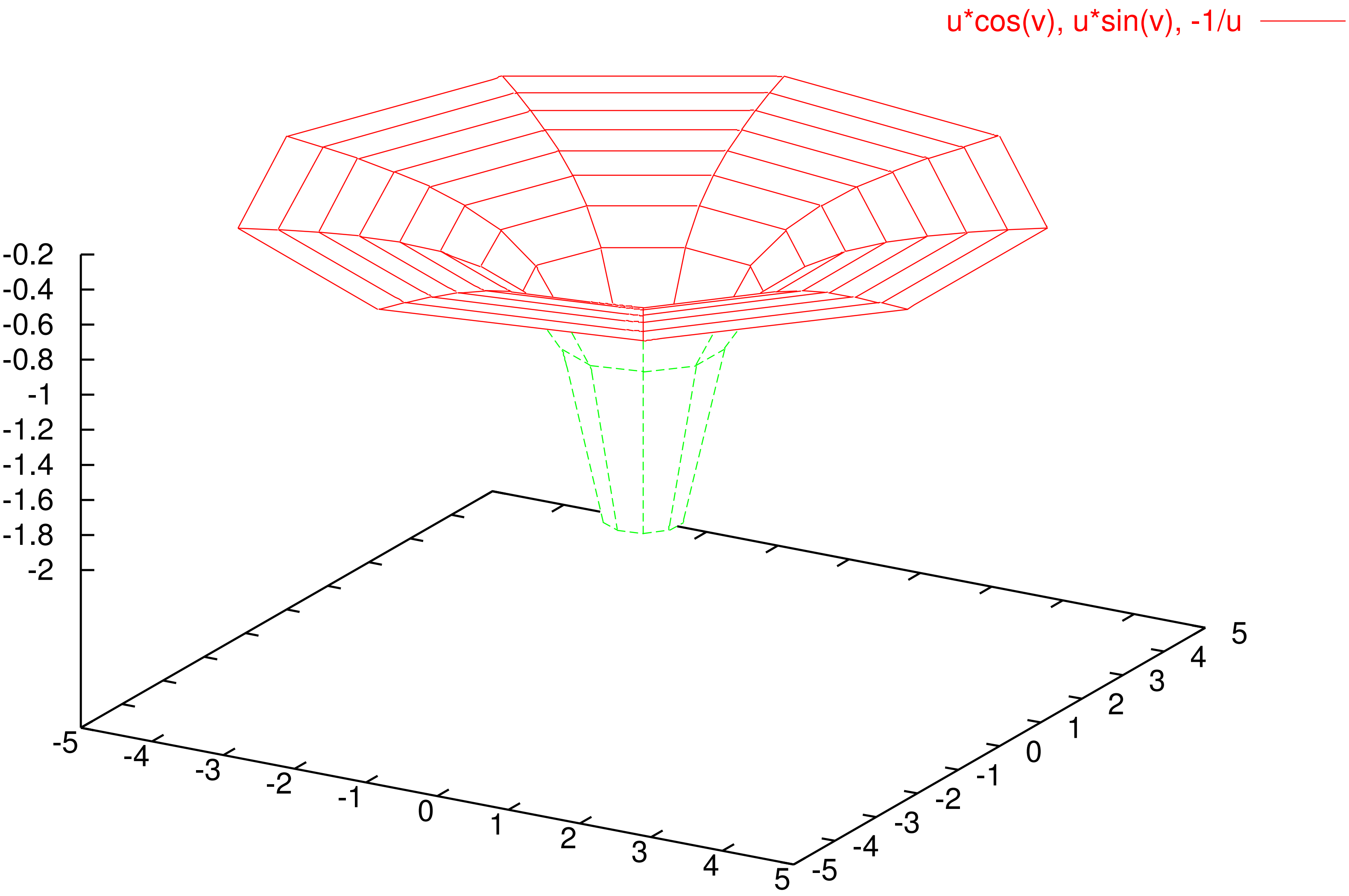
Example

Choose the path shown in the figure. Work done along curved path is zero.

Along radial path, $d\mathbf{r} = dr\hat{\mathbf{r}}$.

$$\begin{aligned} U(\mathbf{r}) &= -\int \mathbf{F}(\mathbf{r}) \cdot d\mathbf{r} \\ &= k\int_{r_0}^{r} \frac{dr}{r^2} \\ &= k\left(-\frac{1}{r} + \frac{1}{r_0}\right) \\ &= -\frac{k}{r} \qquad r_0 \longrightarrow \infty \end{aligned}$$

Example

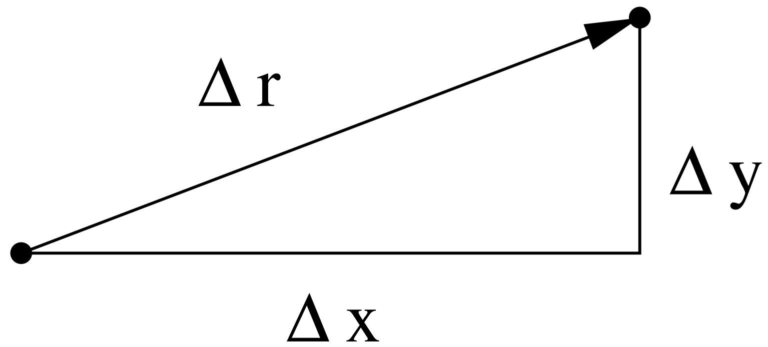


u*cos(v), u*sin(v), -1/u
-0.2
-0.4
-0.6
-0.8
-1
-1.2
-1.4
-1.6
-1.8
-2
-5
-4
-3
-2
-1
0
1
2
3
4
5
-5
-4
-3
-2
-1
0
1
2
3
4
5

Potential Energy

Consider two nearby points $\mathbf{r}$ and $\mathbf{r} + \Delta\mathbf{r}$. The difference in the potential energy is

$$\begin{aligned} \Delta U &= U(\mathbf{r} + \Delta\mathbf{r}) - U(\mathbf{r}) \\ &= \frac{\partial U}{\partial x}\Delta x + \frac{\partial U}{\partial y}\Delta y \end{aligned}$$



Potential Energy and Conservative For

But by definition $\Delta U = -\mathbf{F} \cdot \mathbf{\Delta r} = -F_x \Delta x - F_y \Delta y$. Then

$$F_x = -\frac{\partial U}{\partial x}$$
$$F_y = -\frac{\partial U}{\partial y}$$

In 3-d, clearly $F_z = -\frac{\partial U}{\partial z}$.

Example

Let $U(x,y) = xy$. Then

$$\begin{aligned} F_x &= -\frac{\partial U}{\partial x} = -y \\ F_y &= -\frac{\partial U}{\partial y} = -x \end{aligned}$$

Hence $\mathbf{F} = -y\mathbf{i} - x\mathbf{j}$.

Gradient Operator

Given a potential energy function U,

$$\mathbf{F} = -\mathbf{i}\frac{\partial U}{\partial x} - \mathbf{j}\frac{\partial U}{\partial y} - \mathbf{k}\frac{\partial U}{\partial z}$$

Define an operator that operates on a scalar function and results in a vector as

$$\nabla = \mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}$$

is called Gradient Operator.

Gradient Operator

- The conservative force can be compactly written as

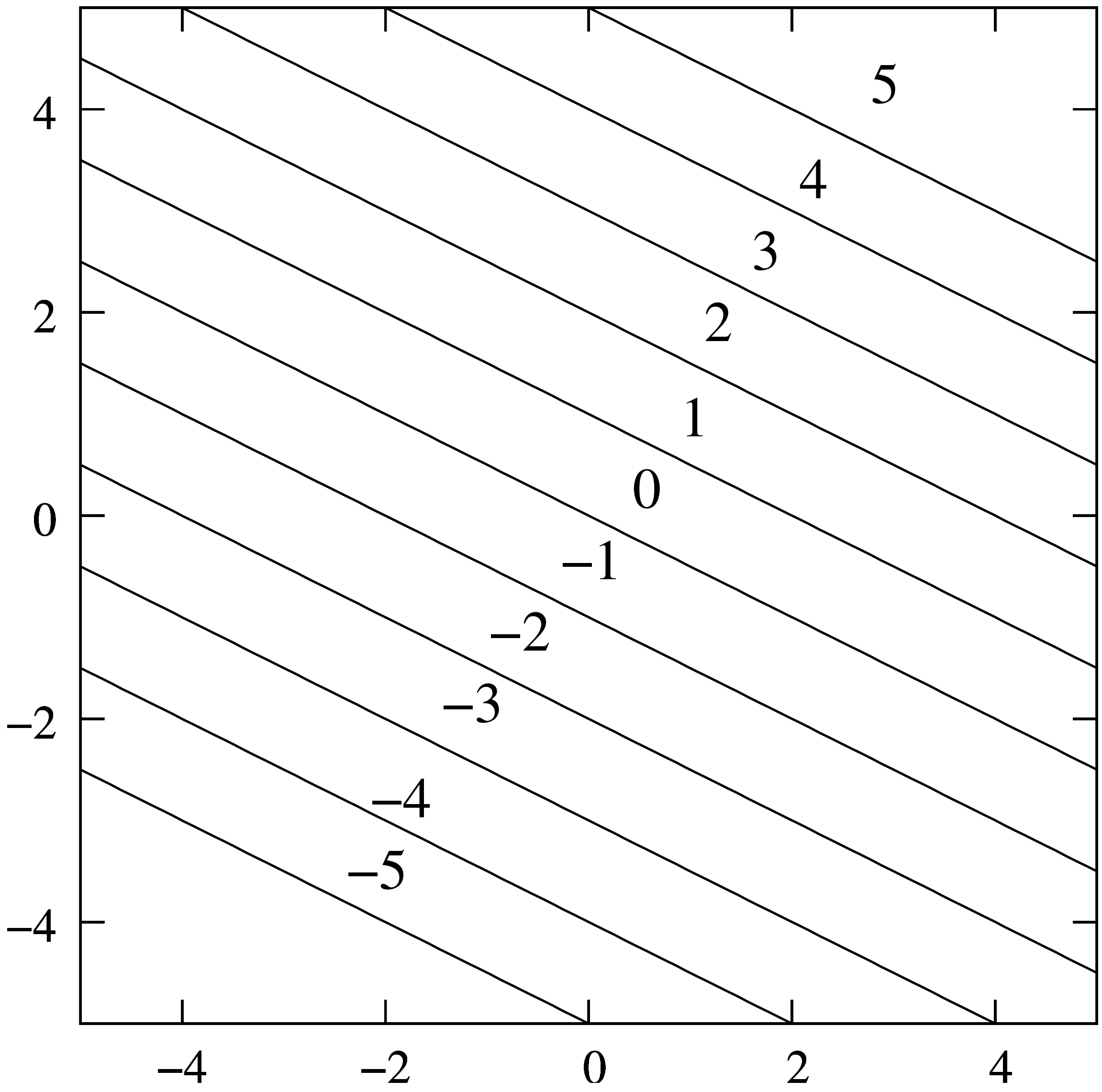
$$\mathbf{F} = -\nabla U$$

- Change in potential energy over a distance $\Delta\mathbf{r}$ is given by

$$\Delta U = -\mathbf{F} \cdot \Delta\mathbf{r} = \nabla U \cdot \Delta\mathbf{r}$$

- If $\mathbf{F}$ is nonzero, U decreases in the direction of the force.
- U increases in the direction of ∇U.
- No change in U if $\Delta\mathbf{r}$ is perpendicular to $\mathbf{F}$.

Constant Energy Surfaces

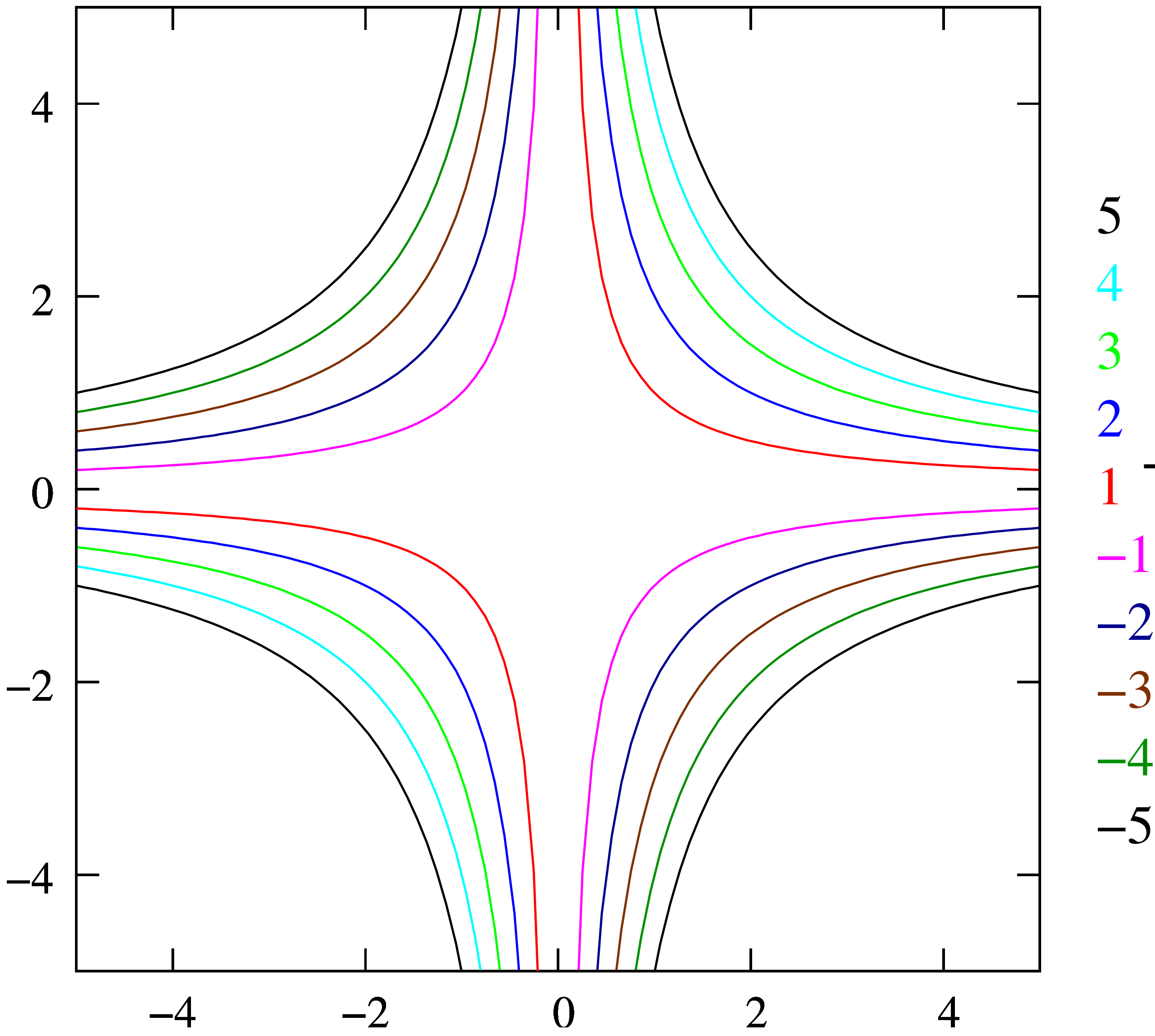


$$U(x, y) = x + 2y$$

Then

$$\begin{aligned} U(x, y) &= c \\ \Rightarrow y &= -x/2 + c \end{aligned}$$

Constant Energy Surfaces

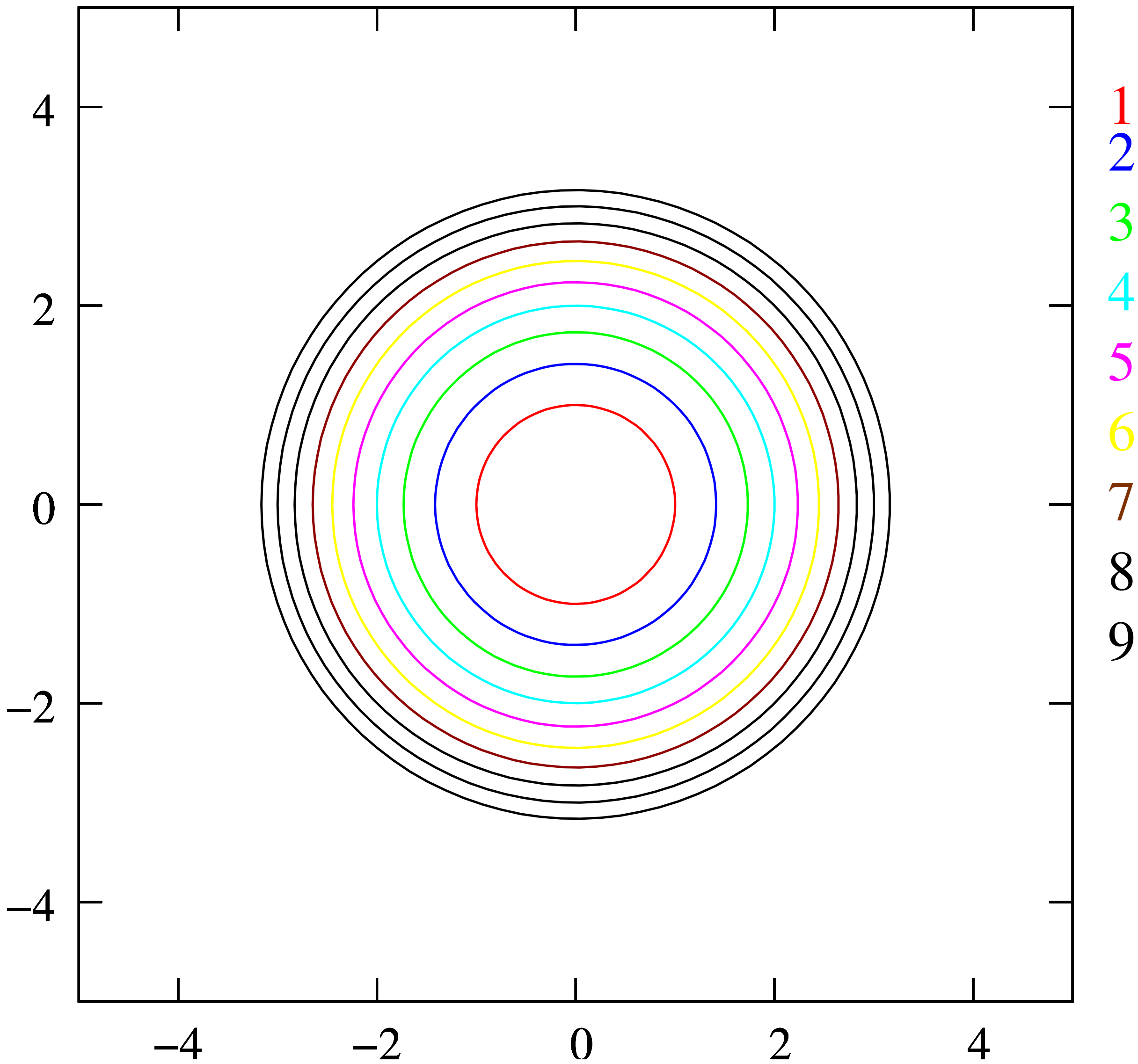


$$U(x,y) = xy$$

Then

$$U(x,y) = c$$
$$\Rightarrow y = c/x$$

Constant Energy Surfaces

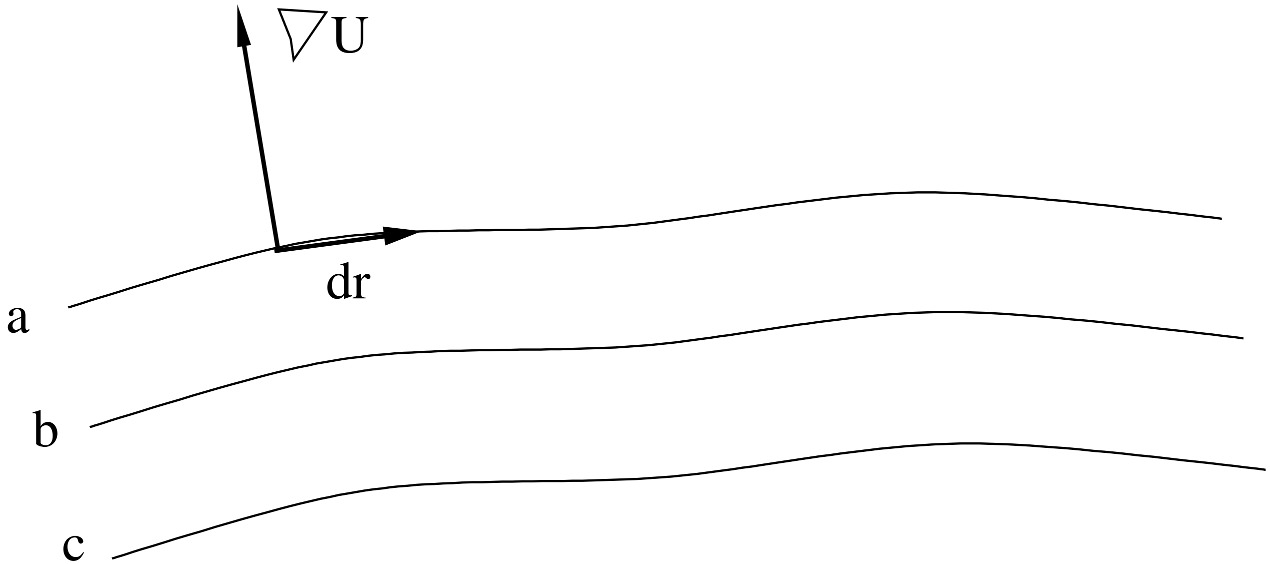


$$U(x, y) = x^2 + y^2$$

Then

$$\begin{aligned} U(x, y) &= c \\ \Rightarrow x^2 + y^2 &= c \end{aligned}$$

Gradient



The gradient of a potential energy surface is perpedicular to the constant energy surface.

Conservative forces

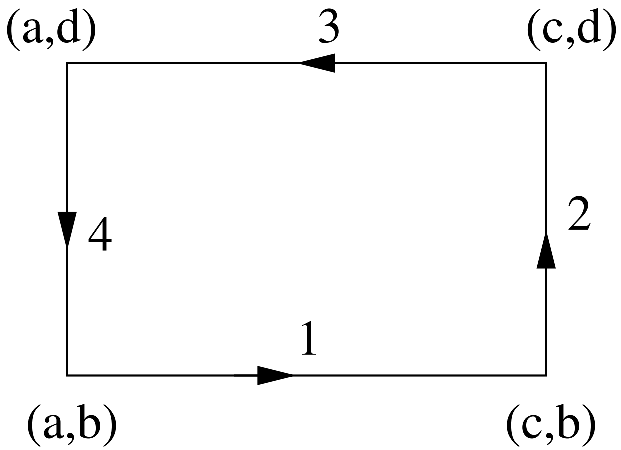
Consider a force field given by a vector function

$\mathbf{F} = F_x(x,y)\,\mathbf{i} + F_y(x,y)\,\mathbf{j}$. The figure show a rectangular closed path.

Line integral of $\mathbf{F}$ along horizontal paths

$$\int_1 \mathbf{F} \cdot d\mathbf{r} + \int_3 \mathbf{F} \cdot d\mathbf{r}$$

$$= \int_a^c F_x(x,b)\, dx - \int_a^c F_x(x,d)\, dx$$



Conservative Forces

Now consider

$$
\begin{aligned}
& \int_a^c \int_b^d \left(\frac{\partial F_x}{\partial y} \right) dx \, dy \\
= \; & \int_a^c dx \left(\int_b^d \left(\frac{\partial F_x}{\partial y} \right) dy \right) \\
= \; & \int_a^c dx \left(F_x \left(x, d \right) - F_x(x, b) \right) \\
= \; & - \int_1 \mathbf{F} \cdot d\mathbf{r} - \int_3 \mathbf{F} \cdot d\mathbf{r}
\end{aligned}
$$

Conservative Forces

Simillarly

$$\int_a^c \int_b^d \left(\frac{\partial F_y}{\partial x} \right) dx\, dy$$

$$= \int_2 \mathbf{F} \cdot d\mathbf{r} + \int_4 \mathbf{F} \cdot d\mathbf{r}$$

Conservative Forces

Hence

$$\int_C \mathbf{F} \cdot d\mathbf{r} = \int_a^c \int_b^d \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) dx dy$$

For a conservative force, the integral over closed loop must be zero.

A force is conservative if and only if

$$\frac{\partial F_y}{\partial x} = \frac{\partial F_x}{\partial y}$$

or

$$\left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) = 0$$

Curl

This condition can be written more compact way. Using the gradient operator, define following operation as

$$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

The z-component of this is given by

$$(\nabla \times \mathbf{F})_z = \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$

In three dimensions, a force is conservative if and only if curl of f is zero.