Physics I

Lecture 4

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Kinetic Energy

Kinetic Energy of a particle in motion is defined as

$$KE = \frac{1}{2}mv^2$$

where v is the instantaneous velocity of the particle.

In 2D, clearly

$$\begin{aligned} \frac{1}{2}mv^2 &= \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} \\ &= \frac{1}{2}m(v_x^2 + v_y^2) \end{aligned}$$

Kinetic Energy changes as particle moves.

Example

A particle is moving on a circular path in horizontal plane. $R = 1$ m, $m = 1$ kg. $\mu = 0.3$. At $t = 0$, $\theta_0 = 0$, $\omega_0 = 9$ rad/sec.

The retarding acceleration $\alpha = \mu g = 3$ rad/sec^2.

Velocity is given by $R(\omega_0 - \alpha t) = (9 - 3t)$ m/s.

Kinetic Energy is then

$$
\begin{aligned}
KE(t) &= \frac{1}{2}m(v(t))^2 \\
&= \frac{1}{2}(9 - 3t)^2 \text{ J}
\end{aligned}
$$

Kinetic energy diminishes.

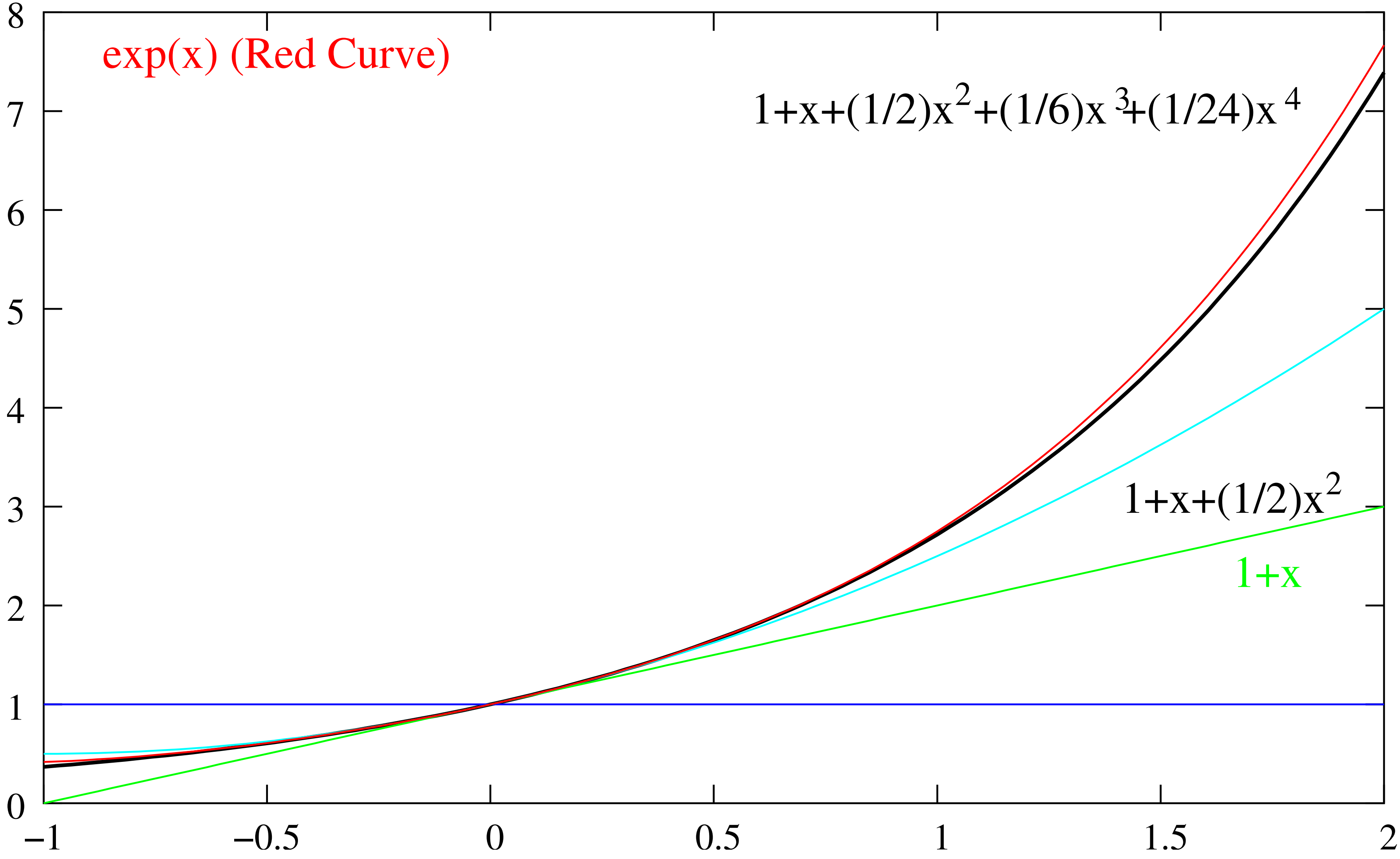
Taylor Series

Let $f(x)$ be differentiable to any order. The Taylor theorem is

$$f(x+\delta x) = f(x) + f'(x)\delta x + \frac{1}{2!}f''(x)(\delta x)^2 + \cdots + R_n(x)$$

- If the series is convergent, the remainder $R_n(x)$ vanishes as $n \to \infty$
- The function f can be approximated in the vicinity of x by polynomial in δx if δx is sufficiently small.

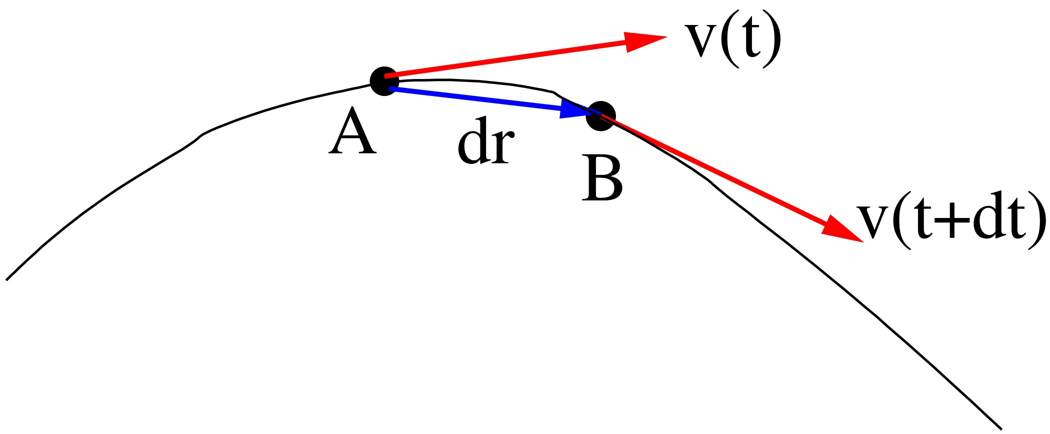
Taylor Series



Work

A particle moves a little distance $\Delta\mathbf{r}$ in time Δt under influence of force $\mathbf{F}$. Say its velocity changes from $\mathbf{v}(t)$ to $\mathbf{v}(t+\Delta t)$. Since

$$\mathbf{v}(t+\Delta t) = \mathbf{v}(t) + \frac{d\mathbf{v}}{dt}\Delta t + O(2)$$



Work

Then, the change in the kinetic energy is

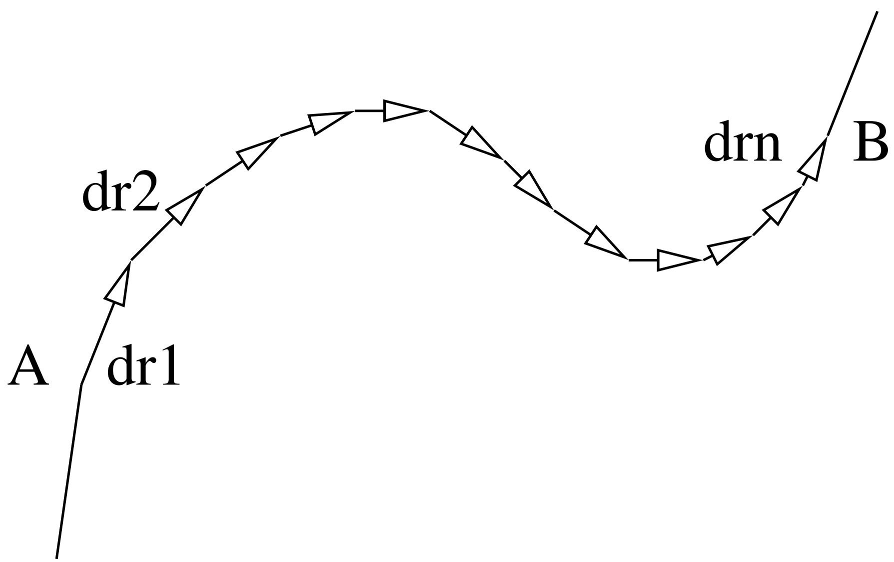
$$
\begin{aligned}
& \frac{1}{2}m\mathbf{v}\left(t+\Delta t\right)\cdot\mathbf{v}\left(t+\Delta t\right)-\frac{1}{2}m\mathbf{v}\left(t\right)\cdot\mathbf{v}\left(t\right) \\
= \quad & \frac{1}{2}m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}\,\Delta t+\frac{1}{2}m\mathbf{v}\cdot\frac{d\mathbf{v}}{dt}\,\Delta t+O\left(2\right) \\
= \quad & \mathbf{F}\cdot\Delta\mathbf{r}
\end{aligned}
$$

The quantity $\mathbf{F}\cdot\Delta\mathbf{r}$ is called the work done by force $\mathbf{F}$ on the particle.

Work

Work done by the force F in moving the particle from position A to position B *along a given path* is given by

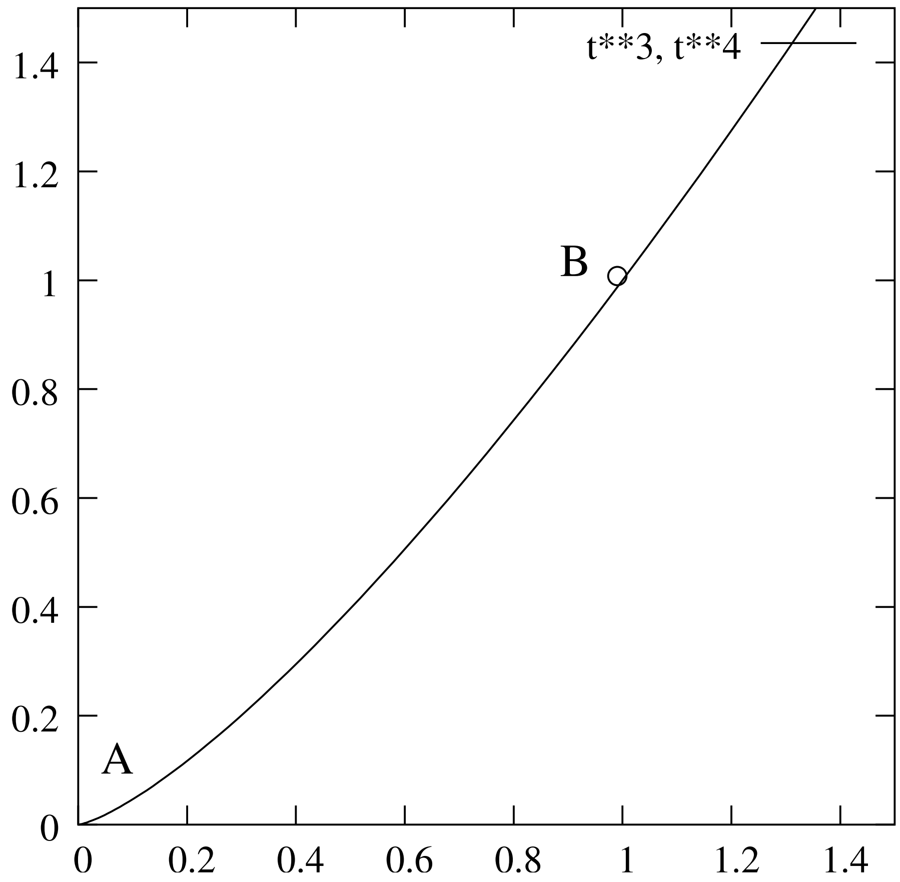
$$W_{AB} = \sum \mathbf{F}_i \cdot \Delta \mathbf{r}_i = \int_A^B \mathbf{F} \cdot d\mathbf{r}$$



Example

A particle (1 kg) starts from origin with zero velocity under the influence of force $\mathbf{F} = 6t\hat{\mathbf{i}} + 12t^2\hat{\mathbf{j}}$. (SI units.) Then acceleration, velocity and position are

$$\begin{aligned} \mathbf{a} &= 6t\hat{\mathbf{i}} + 12t^2\hat{\mathbf{j}} \\ \mathbf{v} &= 3t^2\hat{\mathbf{i}} + 4t^3\hat{\mathbf{j}} \\ \mathbf{r} &= t^3\hat{\mathbf{i}} + t^4\hat{\mathbf{j}} \end{aligned}$$



Example Continued...

At time t, In time dt, suppose that the particle moves by $d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}}$.
Since $x = t^3$, clearly $dx = 3t^2 dt$ and simillarly $dy = 4t^3 dt$.

$$d\mathbf{r} = \left(3t^2\hat{\mathbf{i}} + 4t^3\hat{\mathbf{j}}\right) dt$$

Then the work done by the force

$$\begin{aligned} W_{AB} &= \int_A^B \mathbf{F} \cdot d\mathbf{r} \\ &= \int_0^1 \left(18t^3 + 48t^5\right) dt \\ &= \frac{25}{2} \end{aligned}$$

Properties of Work

- If there are two forces, say $\mathbf{F}_1$ and $\mathbf{F}_2$ acting on a particle, net work done on a particle is, sum of work done by individual forces.

$$
\begin{aligned}
W_{AB} &= \int_A^B (\mathbf{F}_1 + \mathbf{F}_2) \cdot d\mathbf{r} \\
&= \int_A^B \mathbf{F}_1 \cdot d\mathbf{r} + \int_A^B \mathbf{F}_2 \cdot d\mathbf{r} \\
&= W_{AB}^1 + W_{AB}^2
\end{aligned}
$$

Properties of Work

- If a particle is moved from A to B and then B to C. Then

$$\begin{aligned} W_{AC} &= \int_A^C \mathbf{F} \cdot d\mathbf{r} \\ &= \int_A^B \mathbf{F} \cdot d\mathbf{r} + \int_B^C \mathbf{F} \cdot d\mathbf{r} \\ &= W_{AB} + W_{BC} \end{aligned}$$

- $W_{AB} = -W_{BA}$

Work-Energy Theorem

In each time period Δt, the particle moves a distance $\Delta \mathbf{r}$. Then the change in Kinetic Energy ΔK is related to

$$\Delta K = \mathbf{F} \cdot \Delta \mathbf{r}$$

When particle moves from position A to B, all small contributions add to give

$$\begin{aligned} \frac{1}{2} m v_B^2 - \frac{1}{2} m v_A^2 &= \int_A^B \mathbf{F} \cdot d\mathbf{r} \\ &= W_{AB} \end{aligned}$$

Work-Energy Theorem

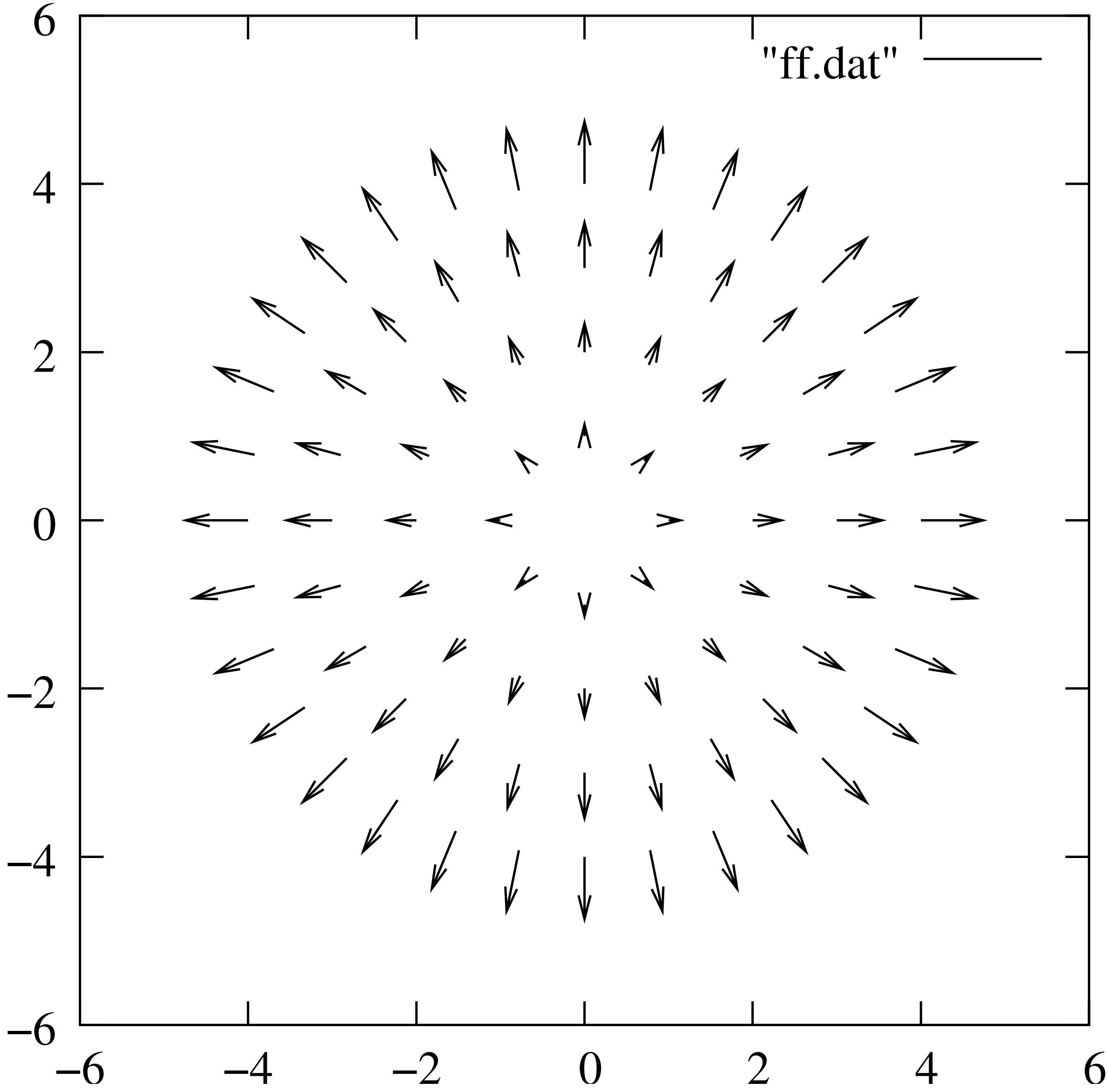
In case, there are several forces acting on the particle,

$$\begin{aligned}\frac{1}{2}mv_B^2 - \frac{1}{2}mv_A^2 &= \sum_i \int_A^B \mathbf{F}_i \cdot d\mathbf{r} \\ &= \sum_i W_{AB}^i\end{aligned}$$

Force Fields

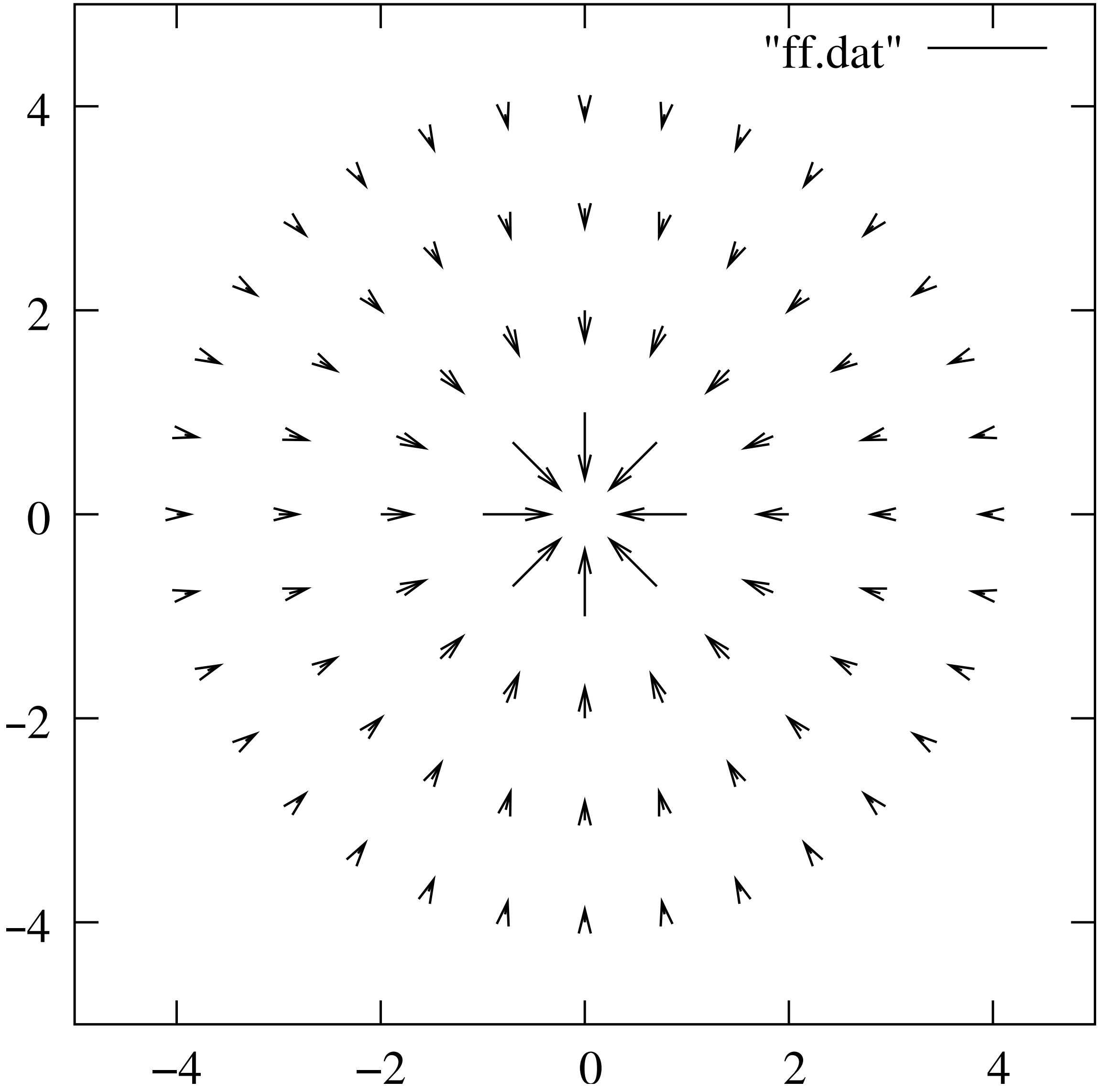
- If a force is a function of position vector only and is defined at all points of the space, then such forces will be called force fields.
- The force fields do not explicitly depend on time.
- The value of the force on a particle at a point doesnot depend on the history of the particle.

Force Fields: Examples



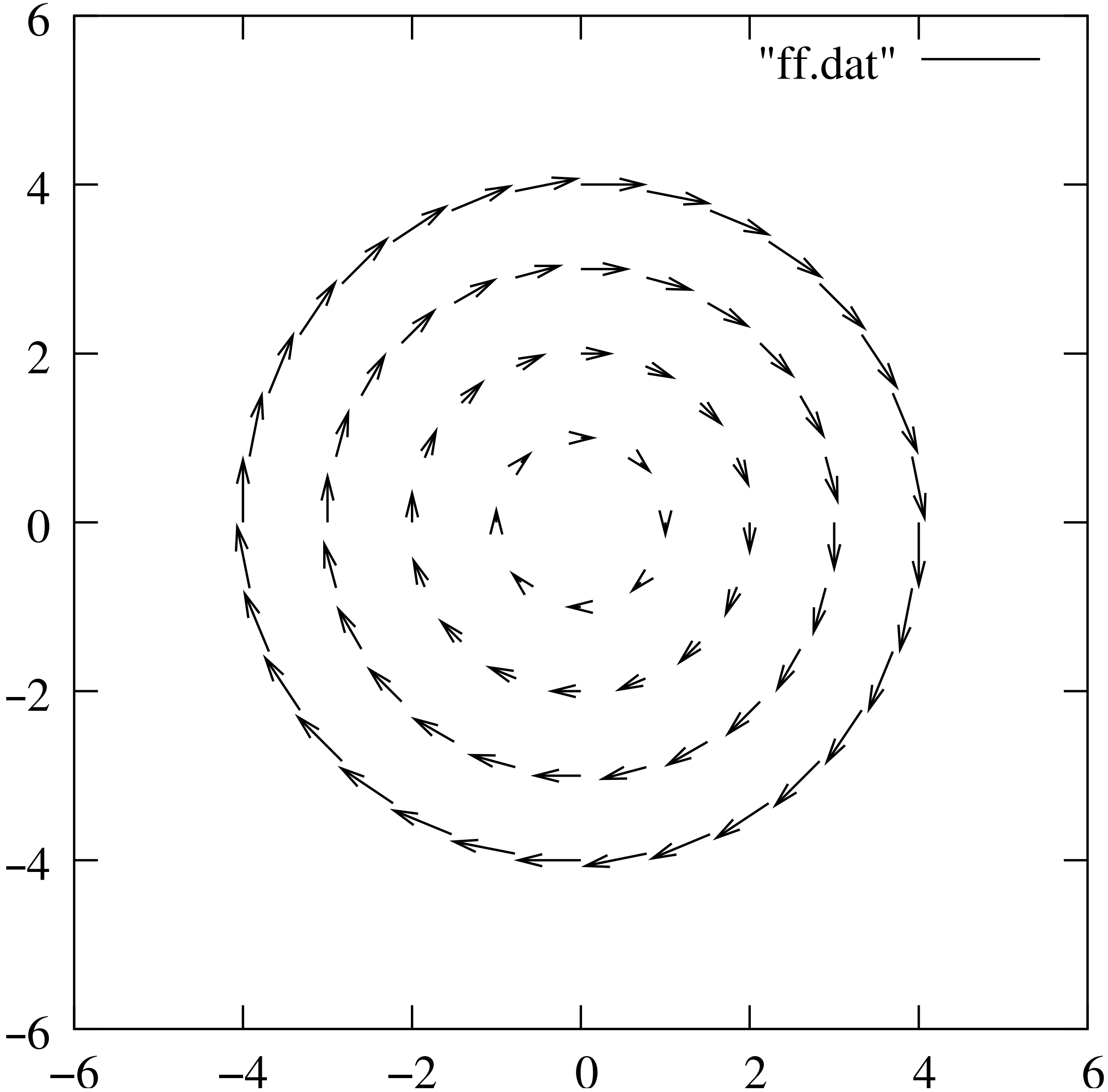
$$\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$$

Force Fields: Examples



$$\mathbf{F} = \frac{1}{r^2}\hat{\mathbf{r}}$$

Force Fields: Examples

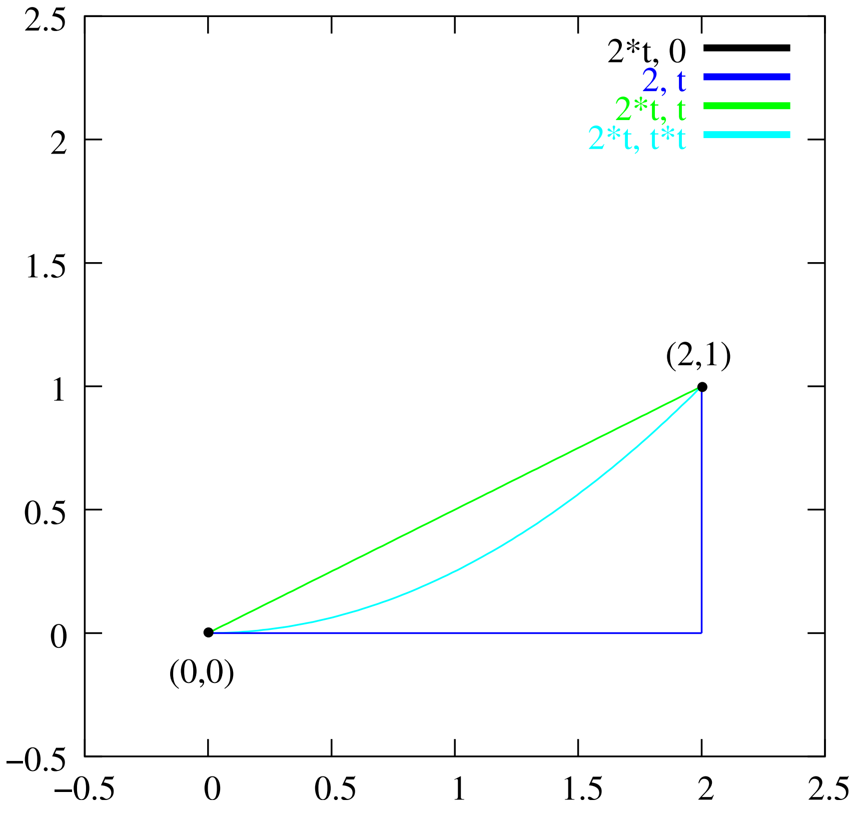


$$\mathbf{F} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$$

Work Done by Force Fields

Let $\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

A particle moves from origin to (2,1). Some possible paths are shown in the figure.



Examples Continued...

Given $\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

Path 1: $(0,0) \longrightarrow (2,0) \longrightarrow (2,1)$

Along $(0,0) \longrightarrow (2,0)$, $d\mathbf{r} = dx\hat{\mathbf{i}}$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_0^2 F_x dx = 2$$

Along $(2,0) \longrightarrow (2,1)$, $d\mathbf{r} = dy\hat{\mathbf{j}}$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_0^1 F_y dy = 1/2$$

Hence work done by $\mathbf{F}$ along path 1 is

$$W_{AB} = 2 + 1/2 = 2.5$$

Examples Continued...

Given $\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

Path 2: $y = \frac{1}{2}x$

Along path 2, $dy = 1/2dx$ and $d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} = (\hat{\mathbf{i}} + 1/2\hat{\mathbf{j}})dx$

$$\begin{aligned}\int \mathbf{F}\cdot d\mathbf{r} &= \int_0^2 \mathbf{F}\cdot(\hat{\mathbf{i}} + 1/2\hat{\mathbf{j}})dx \\ &= \int_0^2 (x + (x/2)(1/2))dx \\ &= 2.5\end{aligned}$$

Hence work done by $\mathbf{F}$ along path 2 is

$$W_{AB} = 2.5$$

Examples Continued...

Given $\mathbf{F} = x\hat{\mathbf{i}} + y\hat{\mathbf{j}}$.

Path 3: $y = \frac{1}{4}x^2$

Along path 3, $dy = \frac{1}{2}x dx$ and $d\mathbf{r} = dx\hat{\mathbf{i}} + dy\hat{\mathbf{j}} = (\hat{\mathbf{i}} + (x/2)\hat{\mathbf{j}})dx$

$$\begin{aligned} \int \mathbf{F} \cdot d\mathbf{r} &= \int_0^2 \mathbf{F} \cdot (\hat{\mathbf{i}} + (x/2)\hat{\mathbf{j}})dx \\ &= \int_0^2 (x + (x^2/4)(x/2))dx \\ &= 2.5 \end{aligned}$$

Hence work done by $\mathbf{F}$ along path 3 is

$$W_{AB} = 2.5$$

Another Example

Given $\mathbf{F} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$.

Path 1: $(0,0) \longrightarrow (1,0) \longrightarrow (1,1)$

Along $(0,0) \longrightarrow (1,0)$, $d\mathbf{r} = dx\hat{\mathbf{i}}$ and $\mathbf{F} = -x\hat{\mathbf{j}}$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_0^1 F_x dx = 0$$

Along $(1,0) \longrightarrow (1,1)$, $d\mathbf{r} = dy\hat{\mathbf{j}}$ and $\mathbf{F} = y\hat{\mathbf{i}} - \hat{\mathbf{j}}$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_0^1 F_y dy = -1$$

Hence work done by $\mathbf{F}$ along path 1 is

$$W_{AB} = 0 - 1 = -1$$

Another Example

Given $\mathbf{F} = y\hat{\mathbf{i}} - x\hat{\mathbf{j}}$.

Path 2: $(0,0) \longrightarrow (0,1) \longrightarrow (1,1)$

Along $(0,0) \longrightarrow (0,1)$, $d\mathbf{r} = dy\hat{\mathbf{j}}$ and $\mathbf{F} = y\hat{\mathbf{i}}$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_0^1 F_y dy = 0$$

Along $(0,1) \longrightarrow (1,1)$, $d\mathbf{r} = dx\hat{\mathbf{i}}$ and $\mathbf{F} = \hat{\mathbf{i}} - x\hat{\mathbf{j}}$

$$\int \mathbf{F} \cdot d\mathbf{r} = \int_0^1 F_x dx = 1$$

Hence work done by $\mathbf{F}$ along path 1 is

$$W_{AB} = 0 + 1 = 1$$

Work done along path 1 and path 2 are different!