Physics I

Lecture 11

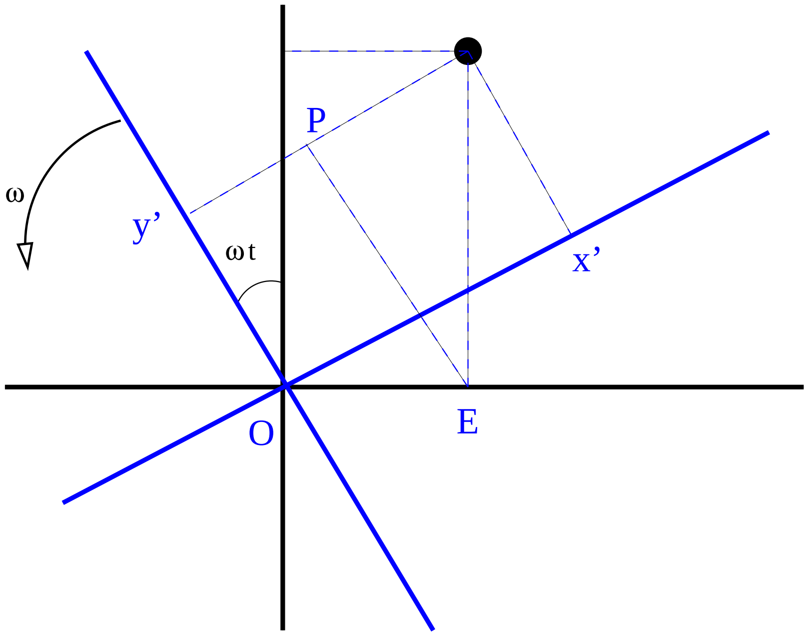
Charudatt Kadolkar

IIT Guwahati

Rotating Coordinate Systems

Transformation Equations

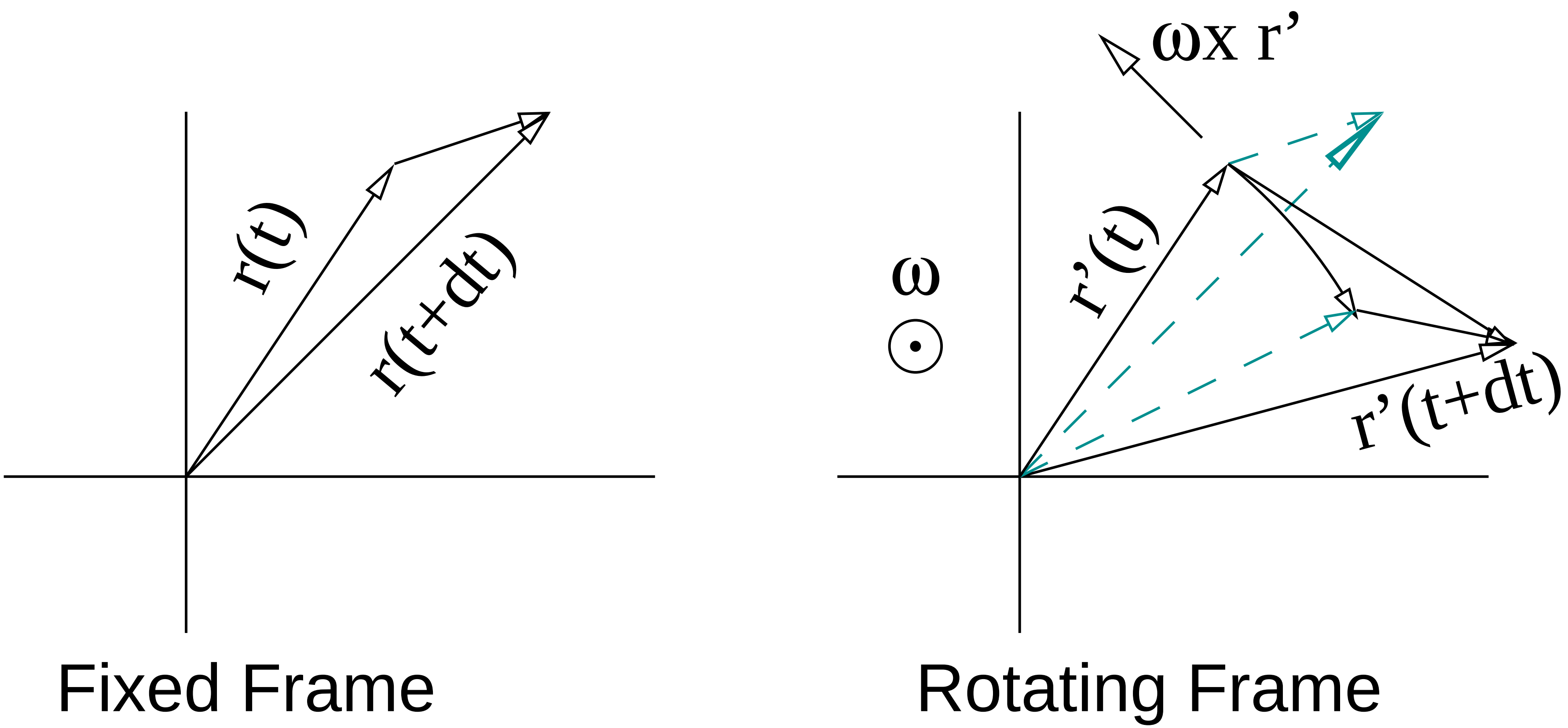
$$\begin{aligned} x' &= x\cos\omega t + y\sin\omega t \\ y' &= -x\sin\omega t + y\cos\omega t \end{aligned}$$



The unit vectors also change with time

$$\begin{aligned} \mathbf{i}' &= \mathbf{i}\cos\omega t - \mathbf{j}\sin\omega t \\ \mathbf{j}' &= \mathbf{i}\sin\omega t + \mathbf{j}\cos\omega t \end{aligned}$$

Rotating Coordinate Systems



$$\vec{u} = \overrightarrow{u'} + \overrightarrow{\omega} \times \overrightarrow{r}$$

Rotating Coordinate Systems

Inverse Transformations

$$
\begin{aligned}
x &= x' \cos\omega t - y' \sin\omega t \\
y &= x' \sin\omega t + y' \cos\omega t
\end{aligned}
$$

Velocity Vector

$$
\begin{aligned}
u_x &= \frac{dx}{dt} \\
&= \frac{dx'}{dt}\cos\omega t - \frac{dy'}{dt}\sin\omega t - x'\omega\sin\omega t - y'\omega\cos\omega t
\end{aligned}
$$

Rotating Coordinate Systems

$$
\begin{aligned}
\mathbf{i}u_x &= u'_x\mathbf{i}\cos\omega t - u'_y\mathbf{i}\sin\omega t - x'\omega\mathbf{i}\sin\omega t - y'\omega\mathbf{i}\cos\omega t \\
\mathbf{j}u_y &= u'_x\mathbf{j}\sin\omega t + u'_y\mathbf{j}\cos\omega t + x'\omega\mathbf{j}\cos\omega t - y'\omega\mathbf{j}\sin\omega t \\
\overrightarrow{u} &= u'_x\mathbf{i}' + u'_y\mathbf{j}' + x'\omega\mathbf{j}' - y'\omega\mathbf{i}' \\
&= \overrightarrow{u'} + \omega x'\left(\mathbf{k}' \times \mathbf{i}'\right) + \omega y'\left(\mathbf{k}' \times \mathbf{j}'\right) \\
&= \overrightarrow{u'} + \overrightarrow{\omega} \times \overrightarrow{r}
\end{aligned}
$$

Rotating Coordinate Systems

Acceleration Vector

$$u_x = \frac{dx'}{dt}\cos\omega t - \frac{dy'}{dt}\sin\omega t - x'\omega\sin\omega t - y'\omega\cos\omega t$$

It turns out to be

$$\overrightarrow{a} = \overrightarrow{a'} + 2\left(\overrightarrow{\omega} \times \overrightarrow{u'}\right) + \overrightarrow{\omega} \times \left(\overrightarrow{\omega} \times \overrightarrow{r'}\right)$$

Rotating Coordinate Systems

If S is inertial then $m\overrightarrow{a} = \overrightarrow{F}$

$$m\overrightarrow{a'} = \overrightarrow{F} - 2m\left(\overrightarrow{\omega} \times \overrightarrow{u'}\right) - m\overrightarrow{\omega} \times \left(\overrightarrow{\omega} \times \overrightarrow{r'}\right)$$

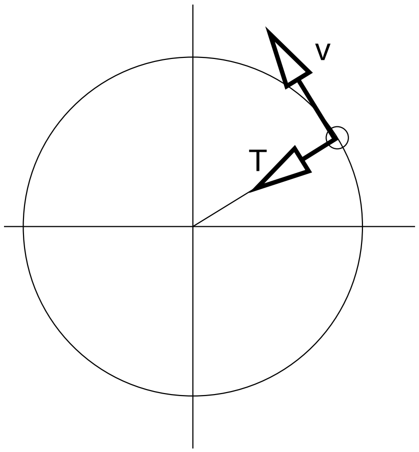
$-2m\left(\overrightarrow{\omega} \times \overrightarrow{u'}\right)$ is called Coriolis Force

$-m\overrightarrow{\omega} \times \left(\overrightarrow{\omega} \times \overrightarrow{r'}\right)$ is called Centrifugal Force

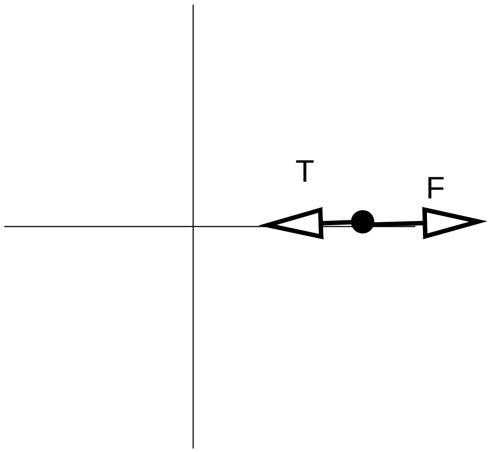
Example 1

Suppose particle is tied by a string and is in uniform circular motion in fixed frame with angular speed ω. In a frame rotating with same angular speed the particle will be at rest. Since $u' = 0$, only pseudo force is centrifugal force, which is $-m\overrightarrow{\omega} \times \left(\overrightarrow{\omega} \times \overrightarrow{r'}\right) = m\omega^2 r'\widehat{\mathbf{r}}'$.

Free body diagrams in two frames are given below.



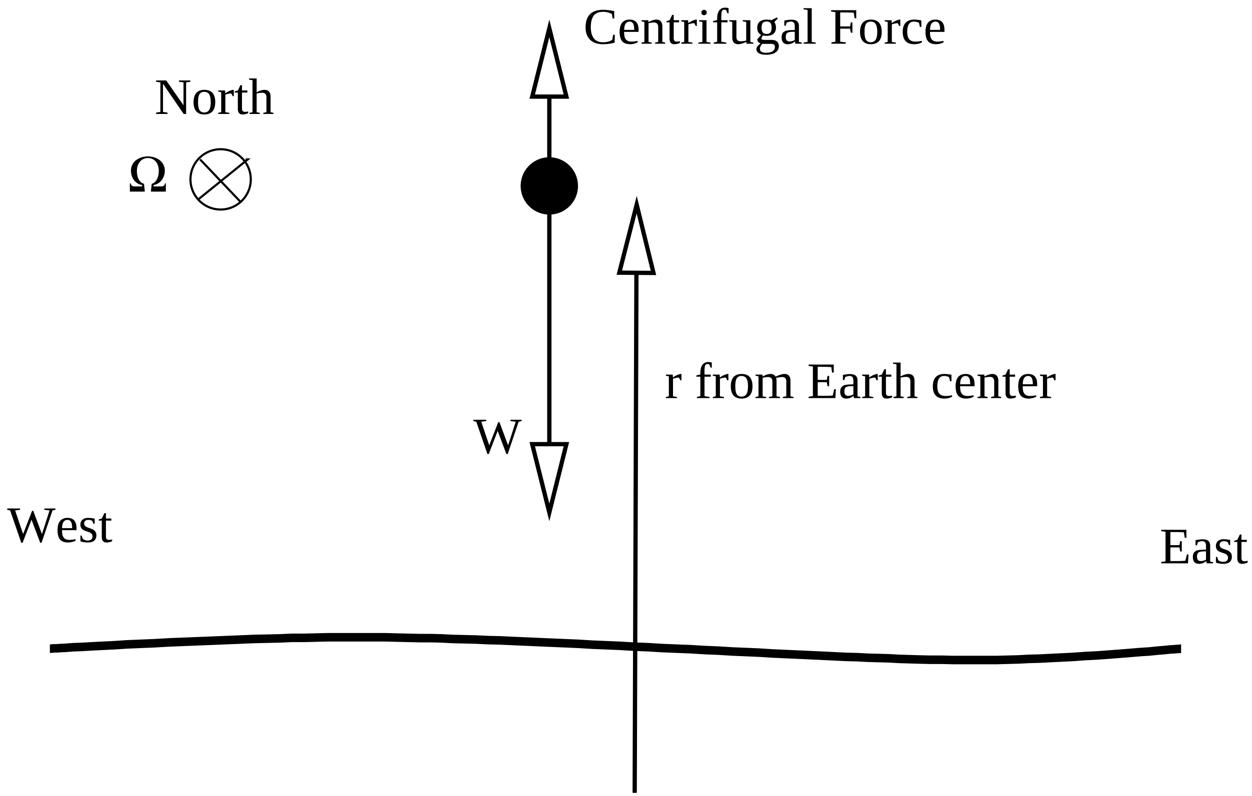
Fixed Frame



Rotating Frame

Example 2

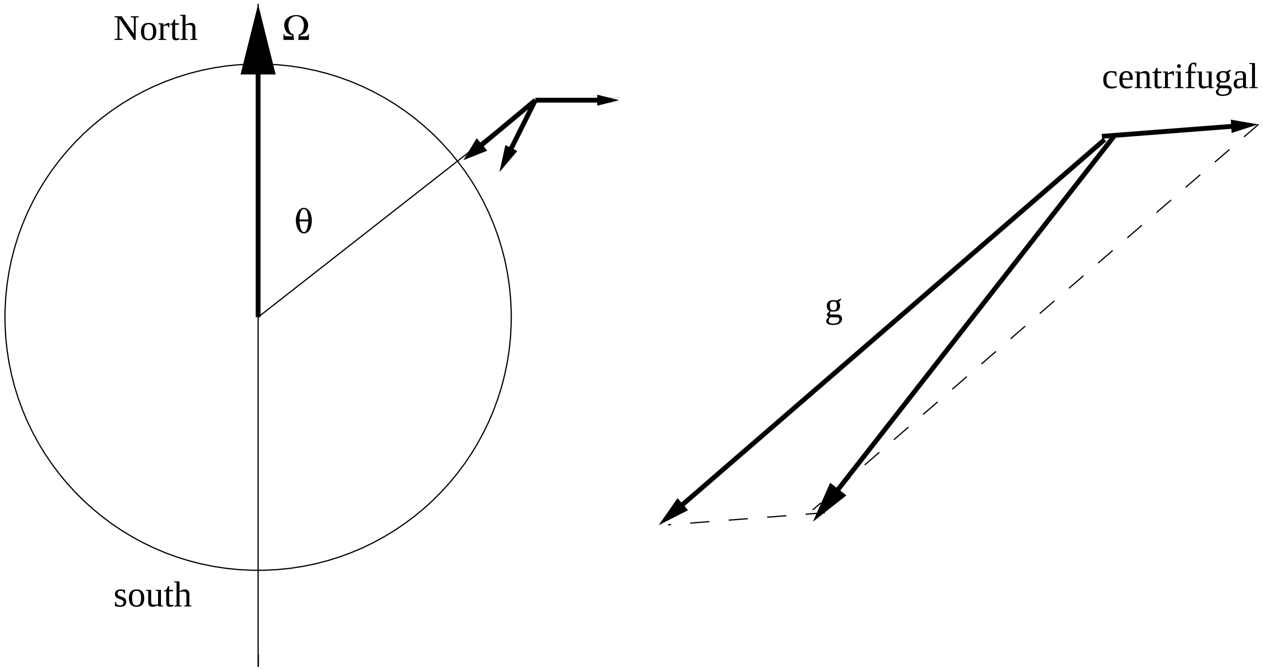
- Angular speed of Earth = 7.27×10^{-5} rad/sec.
- Radius of Earth = 6400 Km = 6.4×10^{6} m



- Centrifugal Acceleration at equator is 3.38×10^{-2} m/s/s

Example 3

At an lattitude θ, what is vertical?



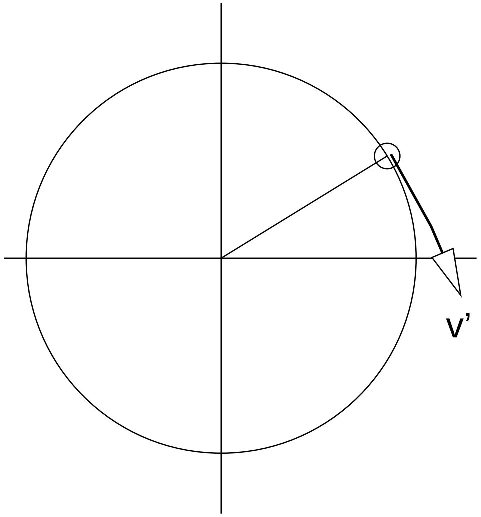
Magnitude of the centrifugal component

$$\Omega^2 R_e \sin\theta$$

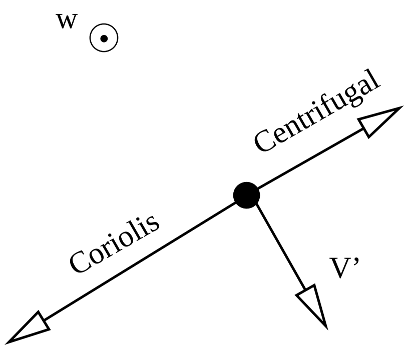
Example 4

Suppose particle is at rest in fixed frame. No Force!

In rotating frame: Circular Trajectory with angular velocity $-\overrightarrow{\omega}$. Centrifugal force is same as before and is $m\omega^2 r'\widehat{\mathbf{r}}'$! But since u' is non-zero, coriolis term is $-2m\omega^2 r'\widehat{\mathbf{r}}'$. So resultant pseudoforce is $-m\omega^2 r'\widehat{\mathbf{r}}'$.



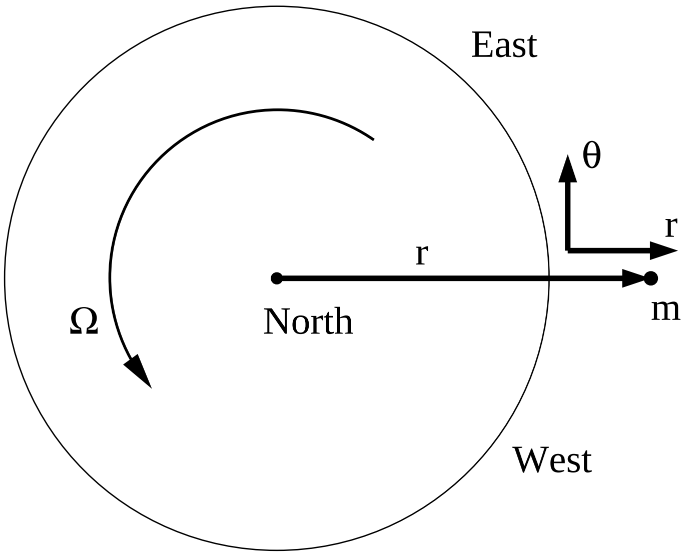
Rotating Frame

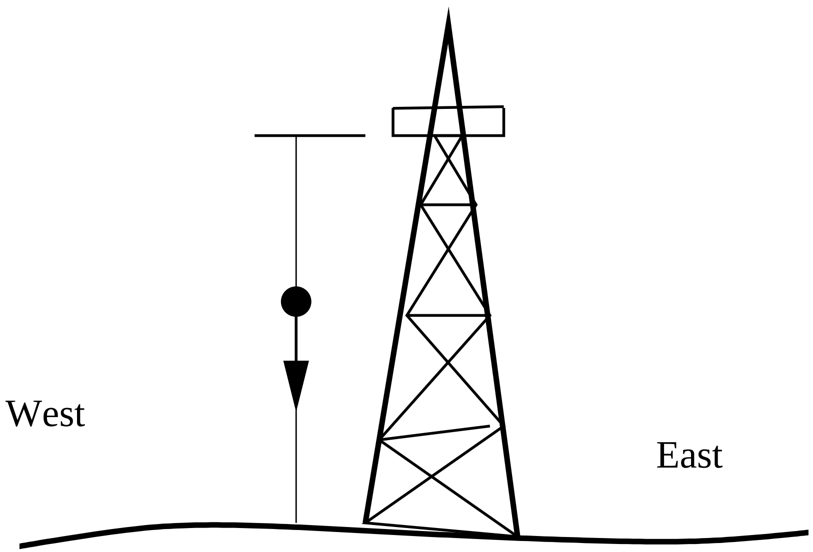


Example 5

A ball is dropped from a tower at equator. (Neglect Air Resistance, Shape of the earth etc.)

Setup a coordinate system in equatorial plane as shown in the figure.





Example Continued...

Equations of Motion are given by

$$\mathbf{F} = -mg\widehat{\mathbf{r}} - 2m\Omega \times \mathbf{v} - m\Omega \times (\Omega \times \mathbf{r})$$

If the velocity of the particle is $\mathbf{v} = \dot{r}\widehat{\mathbf{r}} + r\dot{\theta}\widehat{\theta}$, two components of the force are given by

$$F_r = -mg + 2m\Omega\dot{\theta}r + m\Omega^2 r$$

$$F_\theta = -2m\dot{r}\Omega$$

Example Continued...

Approximations

- Ball falls almost vertically, so the deflection and velocity in horizontal component (θ direction) is very small.
- g is constant, since the ball falls from $R_e + h$ to R_e.

The radial equation is

$$\begin{aligned} m\left(\ddot{r} - mr\dot{\theta}^2\right) &= -mg + 2m\Omega\dot{\theta}r + m\Omega^2 r \\ m\ddot{r} &= -mg + m\Omega^2 r \\ \ddot{r} &= -g \end{aligned}$$

Time taken to fall is $\sqrt{2h/g}$.

Example Continued...

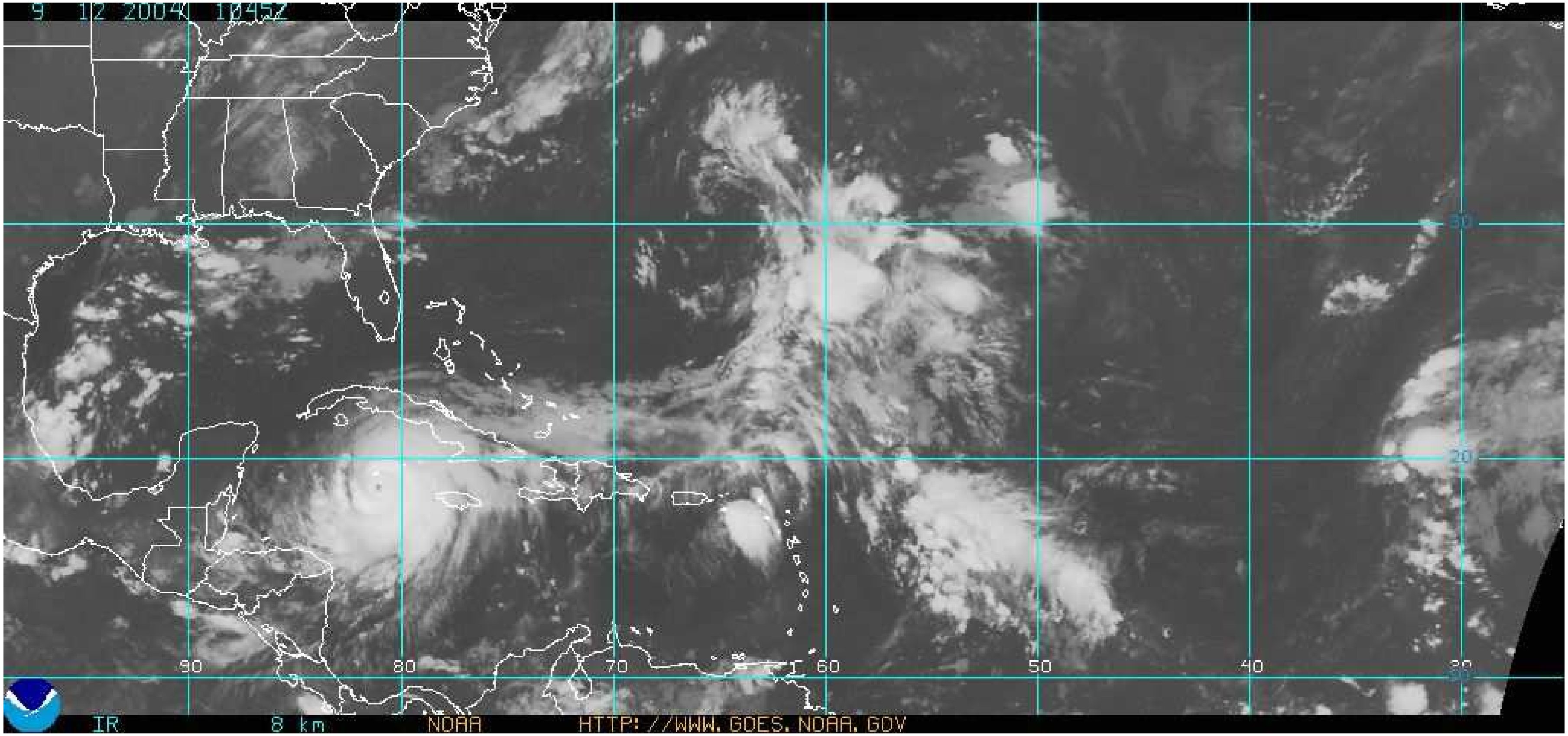
The tangential equation is

$$\begin{aligned} m\left(r\ddot{\theta} + 2m\dot{r}\dot{\theta}\right) &= -2m\dot{r}\Omega \\ R_e\ddot{\theta} &= -2\dot{r}\Omega \\ \ddot{\theta} &= \frac{2g\Omega}{R_e}t \end{aligned}$$

So $\theta = \frac{g\Omega t^3}{3}$. In time T, the deflection to the east will be

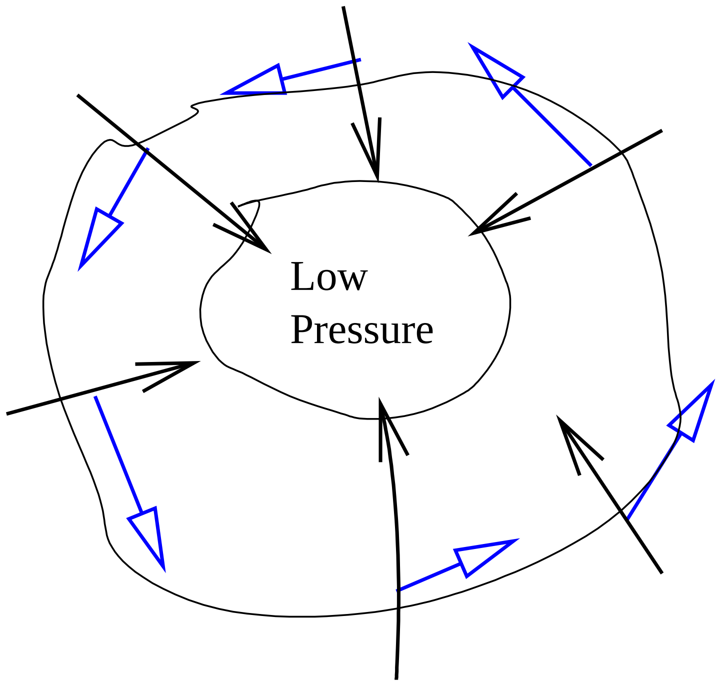
$$y = \frac{1}{3}g\Omega\left(\frac{2h}{g}\right)^{2/3}$$

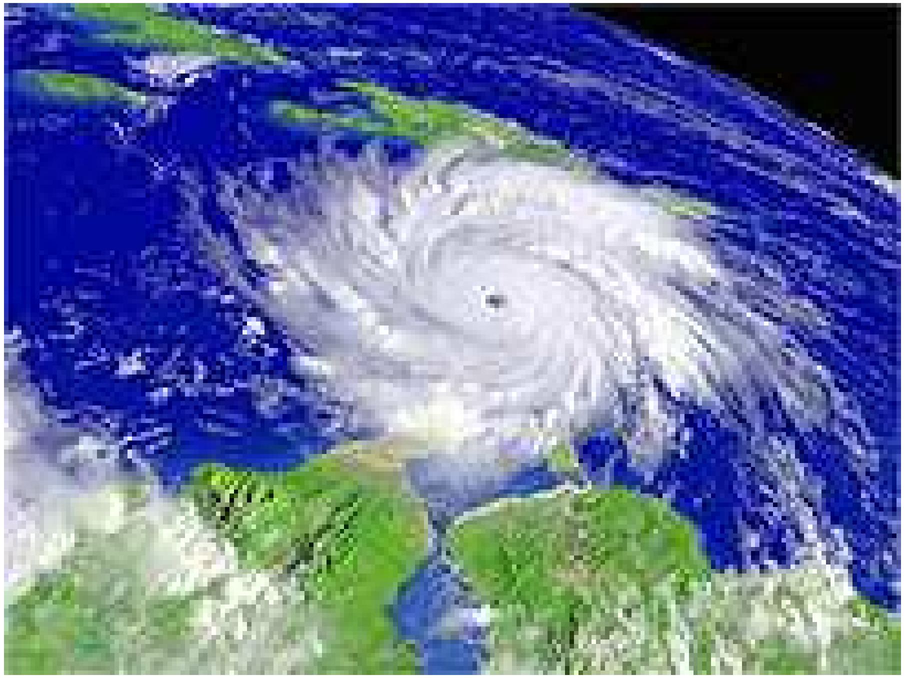
Hurricanes



Hurricane Ivan presently over Jamaica and is heading for Cuba. Picture was taken on 12/9/04.

Hurricanes





As the air moves in to a low pressure area, coriolis force deflects it to right before being sucked in, causing a counter-clockwise spinning motion. In southern hemisphere the spin would be clockwise.