INTRODUCTION TO NONLINEAR DYNAMICS AND STABILITY

Dr. Budhaditya Hazra
Associate Professor,
Structural Engineering,
Department of Civil Engineering,
IIT-Guwahati

INTRODUCTION TO NON-LINEAR BEHAVIOUR

Consider Duffing's Equation

$$\ddot{x} + \omega_0^2 x \pm \alpha x^3 = F \cos \omega t$$

or $$\ddot{x} = -\omega_0^2 x \pm \alpha x^3 + F \cos \omega t \qquad (1)$$

As a first approximation, assume the solution: $$x_1(t) = A \cos \omega t \qquad (2)$$

From (1) & (2) $$\ddot{x}_2 = -A\omega_0^2 \cos \omega t \pm A^3 \alpha \cos^3 \omega t + F \cos \omega t$$

Using $$\cos^3 \omega t = \frac{3}{4} \cos \omega t + \frac{1}{4} \cos 3\omega t$$

$$\ddot{x}_2 = -\left(A\omega_0^2 \pm \frac{3}{4} A^3 \alpha - F\right) \cos \omega t \pm \frac{1}{4} A^3 \alpha \cos 3\omega t$$

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$$\ddot{x}_2 = -(A\omega_0^2 \pm \tfrac{3}{4}A^3\alpha - F)\cos\omega t \pm \tfrac{1}{4}A^3\alpha\cos 3\omega t \qquad (3)$$

By integrating this equation and setting the constants of integration to zero (so as to make the solution harmonic with period $\tau = \frac{2\pi}{\omega}$), we obtain the second approximation:

$$x_2(t) = \frac{1}{\omega^2}(A\omega_0^2 \pm \tfrac{3}{4}A^3\alpha - F)\cos\omega t \pm \frac{A^3\alpha}{36\,\omega^2}\cos 3\omega t \qquad (4)$$

Duffing reasoned at this point that if $x_1(t)$ and $x_2(t)$ are good approximations to the solution $x(t)$, the coefficients of $\cos\omega t$ in the two equations (3) and (4) should not be very different. Thus, by equating these coefficients, we obtain

$$A = \frac{1}{\omega^2}\left(A\omega_0^2 \pm \frac{3}{4}A^3\alpha - F\right)$$

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$$A = \frac{1}{\omega^2}\left(A\omega_0^2 \pm \frac{3}{4}A^3\alpha - F\right)$$

Or,

$$\omega^2 = \omega_0^2 \pm \frac{3}{4}A^2\alpha - \frac{F}{A}$$

For the free vibration of the nonlinear system, $F = 0$

$$\omega^2 = \omega_0^2 \pm \tfrac{3}{4}A^2\alpha \qquad (5)$$

This equation shows that the frequency of the response increases with the amplitude *A* for the hardening spring and decreases for the softening spring

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$$\ddot{x} + c\dot{x} + \omega_0^2 x \pm \alpha x^3 = F \cos \omega t \qquad (6)$$

For a damped system, it was observed in earlier chapters that there is a phase difference between the applied force and the response or solution

Solution: It is more convenient to fix the phase of the solution and keep the phase of the applied force as a quantity to be determined.

Assume that c, A_1, and A_2 are all small, of order α

Assume the first approximation to the solution to be

$$x_1 = \cos \omega t \qquad (7)$$

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Substituting (7) in (6) one obtains

$$\left[(\omega_0^2 - \omega^2)A \pm \frac{3}{4}\alpha A^3\right]\cos \omega t - c\omega A \sin \omega t \pm \frac{\alpha A^3}{4}\cos 3\omega t$$

$$= A_1 \cos \omega t - A_2 \sin \omega t$$

Equating the $\cos \omega t$ and $\sin \omega t$ terms, one obtains:

$$(\omega_0^2 - \omega^2)A \pm \tfrac{3}{4}\alpha A^3 = A_1$$

$$c\omega A = A_2$$

The relation between the amplitude of the applied force and the quantities A and ω can be obtained by squaring and adding the

$$\left[(\omega_0^2 - \omega^2)A \pm \tfrac{3}{4}\alpha A^3\right]^2 + (c\omega A)^2 = A_1^2 + A_2^2 = F^2$$

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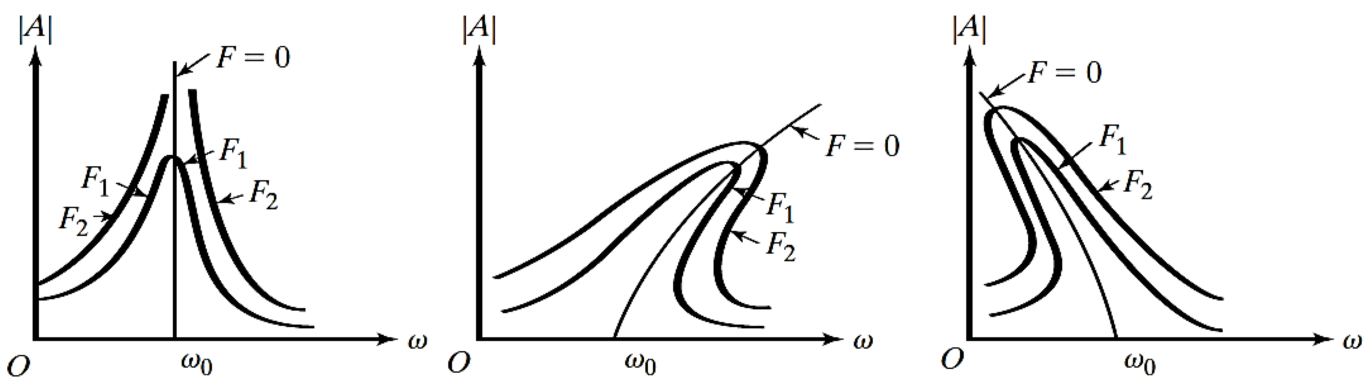
$$\left[(\omega_0^2 - \omega^2)A \pm \tfrac{3}{4}\alpha A^3\right]^2 + (c\omega A)^2 = A_1^2 + A_2^2 = F^2$$

$$S^2(\omega, A) + c^2\omega^2A^2 = F^2$$

$$S(\omega, A) = (\omega_0^2 - \omega^2)A \pm \tfrac{3}{4}\alpha A^3$$



(a) $\alpha = 0$ (b) $\alpha > 0$ (c) $\alpha < 0$

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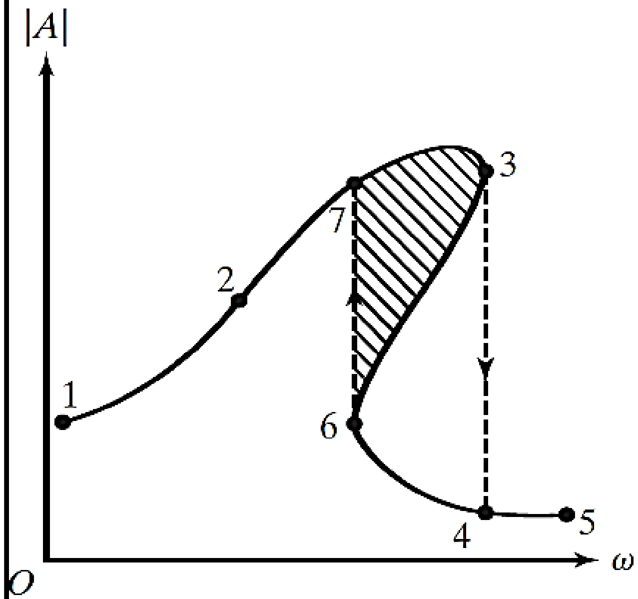


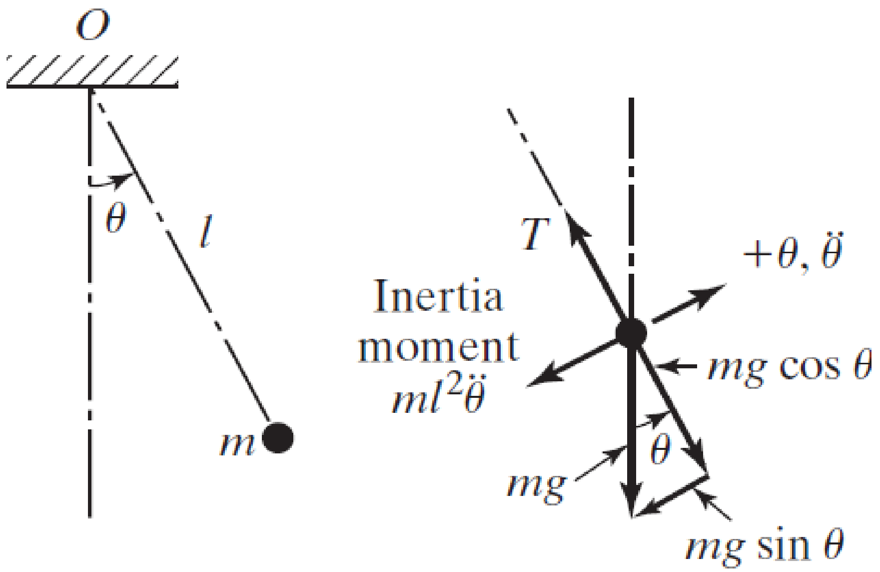
(a) $\alpha > 0$ (hard spring)

This behavior is known as the *jump phenomenon*. It is evident that two amplitudes of vibration exist for a given forcing frequency, as shown in the shaded regions of the curves of Fig. The shaded region can be thought of as unstable in some sense.

Thus an understanding of the jump phenomenon requires a knowledge of the mathematically involved stability analysis of periodic solutions

DIFFERENTIAL EQUATIONS AND DIRECTION FIELDS

Consider undamped pendulum



$$ml^2\ddot{\theta} + mgl\sin\theta = 0$$

$$\ddot{\theta} + \omega_0^2\theta = 0$$

DIFFERENTIAL EQUATIONS AND DIRECTION FIELDS

Consider undamped pendulum

where $\omega_0^2 = g/l$. Introducing $x = \theta$ and $y = \dot{x} = \theta$,

$$\frac{dx}{dt} = y, \qquad \frac{dy}{dt} = -\,\omega_0^2 \sin x$$

or

$$\frac{dy}{dx} = -\frac{\omega_0^2 \sin x}{y}$$

or

$$y\,dy = -\omega_0^2 \sin x\,dx \tag{E.2}$$

Integrating Eq. (E.2) and using the condition that $\dot{x} = 0$ when $x = x_0$ (at the end of the swing), we obtain

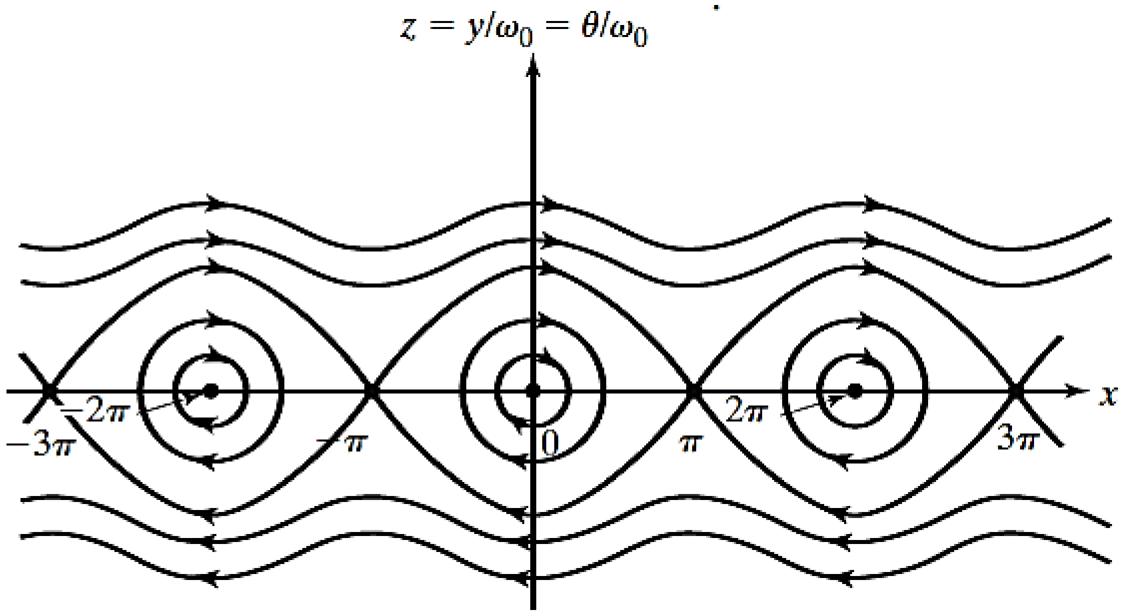
$$y^2 = 2\omega_0^2(\cos x - \cos x_0) \tag{E.3}$$

DIFFERENTIAL EQUATIONS AND DIRECTION FIELDS

Consider undamped pendulum

Introducing $z = y/\omega_0$, Eq. (E.3) can be expressed as

$$z^2 = 2(\cos x - \cos x_0) \tag{E.4}$$



DIFFERENTIAL EQUATIONS AND DIRECTION FIELDS

Consider undamped pendulum

$$y'' = -\sin(y)$$

and the initial conditions

$$y(0) = 1$$
$$y'(0) = 0.$$

Let $y_1 = y$ and $y_2 = y'$, this gives the first order system

$$y_1' = y_2,$$
$$y_2' = -\sin(y_1)$$

DIFFERENTIAL EQUATIONS AND DIRECTION FIELDS

Undamped pendulum

Define an @-function f for the right hand side of the first order system

f = @(t, y) [*expression for* y_1' ; *expression for* y_2'];

$y_1' = y_2$,
$y_2' = -\sin(y_1)$

Here we define f = @(t, y) [y(2);-sin(y(1))]

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Example . Solve the following IVP.

$$\vec{x}' = \begin{pmatrix} 1 & 2 \\ 3 & 2 \end{pmatrix} \vec{x}, \qquad \vec{x}(0) = \begin{pmatrix} 0 \\ -4 \end{pmatrix}$$

Solution

So, the first thing that we need to do is find the eigenvalues for the matrix.

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix}$$

$$= \lambda^2 - 3\lambda - 4$$

$$= (\lambda+1)(\lambda-4) \qquad \Rightarrow \qquad \lambda_1 = -1,\ \lambda_2 = 4$$

Now let's find the eigenvectors for each of these.

$\lambda_1 = -1$:

We'll need to solve,

$$\begin{pmatrix} 2 & 2 \\ 3 & 3 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \Rightarrow \qquad 2\eta_1 + 2\eta_2 = 0 \qquad \Rightarrow \qquad \eta_1 = -\eta_2$$

INTRODUCTION TO STABILITY THEORY

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} -\eta_2 \\ \eta_2 \end{pmatrix} \quad \Rightarrow \quad \vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}, \quad \eta_2 = 1$$

$\lambda_2 = 4$:

We'll need to solve,

$$\begin{pmatrix} -3 & 2 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad \Rightarrow \quad -3\eta_1 + 2\eta_2 = 0 \quad \Rightarrow \quad \eta_1 = \frac{2}{3}\eta_2$$

The eigenvector in this case is,

$$\vec{\eta} = \begin{pmatrix} \frac{2}{3}\eta_2 \\ \eta_2 \end{pmatrix} \quad \Rightarrow \quad \vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}, \quad \eta_2 = 3$$

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Then general solution is then,

$$\vec{x}(t) = c_1 \mathbf{e}^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \mathbf{e}^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

Now, we need to find the constants. To do this we simply need to apply the initial conditions.

$$\begin{pmatrix} 0 \\ -4 \end{pmatrix} = \vec{x}(0) = c_1 \begin{pmatrix} -1 \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

All we need to do now is multiply the constants through and we then get two equations (one for each row) that we can solve for the constants. This gives,

$$\left.\begin{aligned} -c_1 + 2c_2 &= 0 \\ c_1 + 3c_2 &= -4 \end{aligned}\right\} \qquad \Rightarrow \qquad c_1 = -\frac{8}{5}, \; c_2 = -\frac{4}{5}$$

The solution is then,

$$\vec{x}(t) = -\frac{8}{5} \mathbf{e}^{-t} \begin{pmatrix} -1 \\ 1 \end{pmatrix} - \frac{4}{5} \mathbf{e}^{4t} \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

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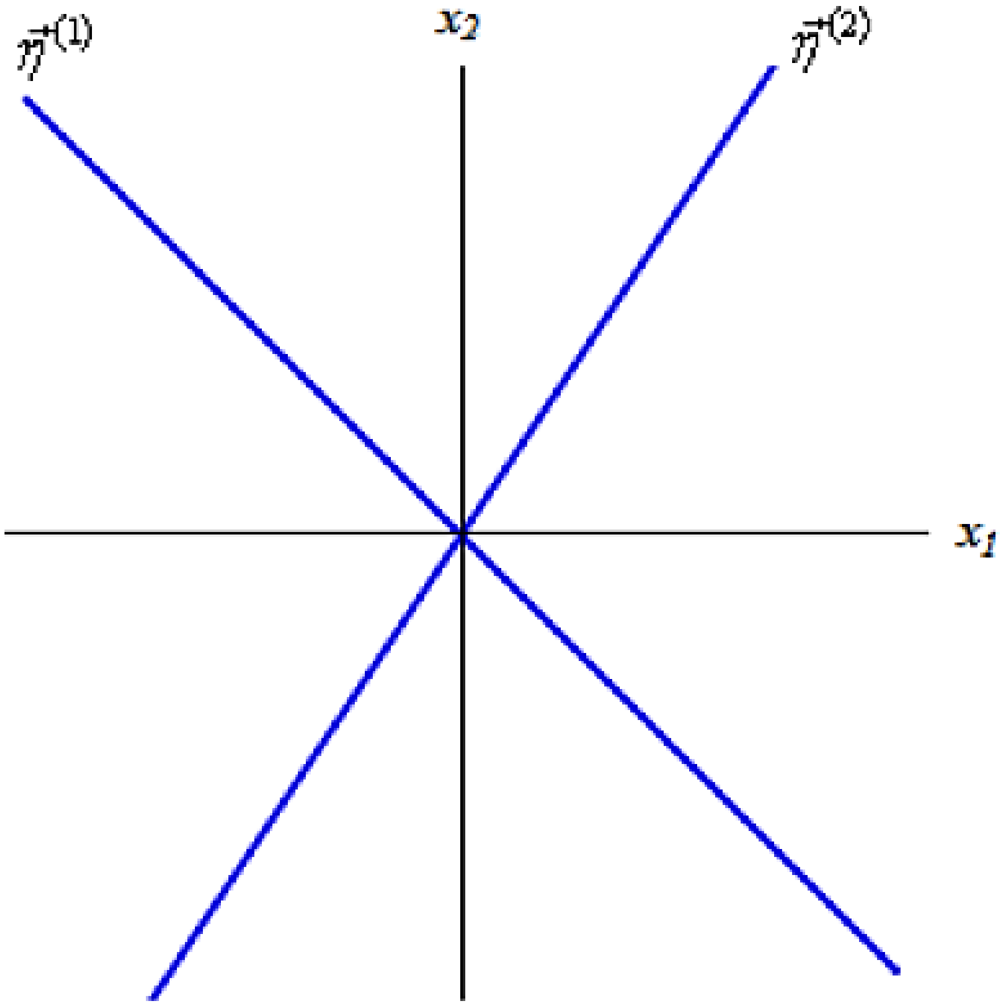
$$\lambda_1 = -1$$

$$\lambda_2 = 4$$

$$\vec{\eta}^{(1)} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

$$\vec{\eta}^{(2)} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$



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- If we have $c_2 = 0$ then the solution = exponential x vector and all that the exponential does is affect the magnitude of the vector and the constant c_1 will affect both the sign and the magnitude of the vector.

- The trajectory in this case will be a straight line that is parallel to the vector, η^1.

- Also notice that as *t* increases the exponential will get smaller and smaller and hence the trajectory will be moving in towards the origin.

- If $c_1 > 0$ the trajectory will be in Quadrant II and if $c_1 < 0$ the trajectory will be in Quadrant IV.

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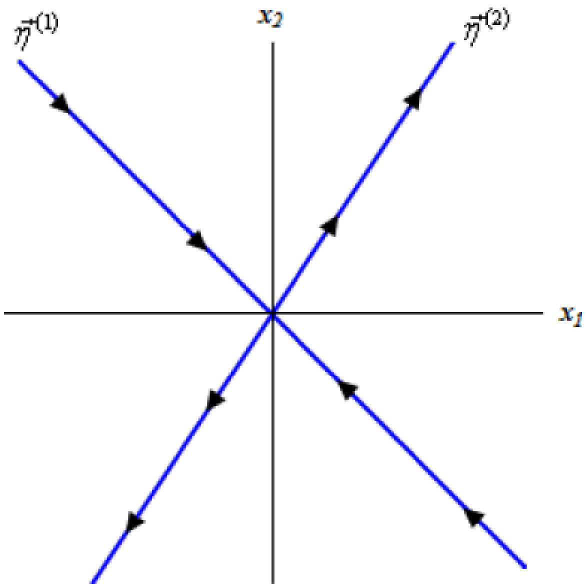

Eigenvalues that are negative will correspond to solutions that will move towards the origin as *t* increases in a direction that is parallel to its eigenvector.

Likewise, eigenvalues that are positive move away from the origin as *t* increases in a direction that will be parallel to its eigenvector.

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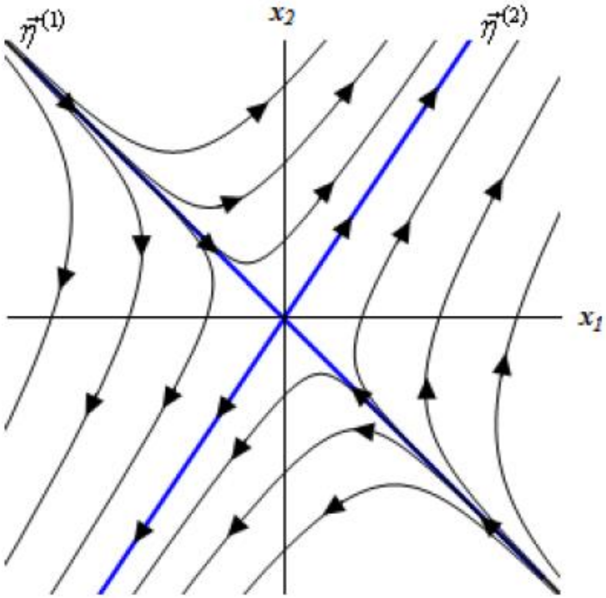


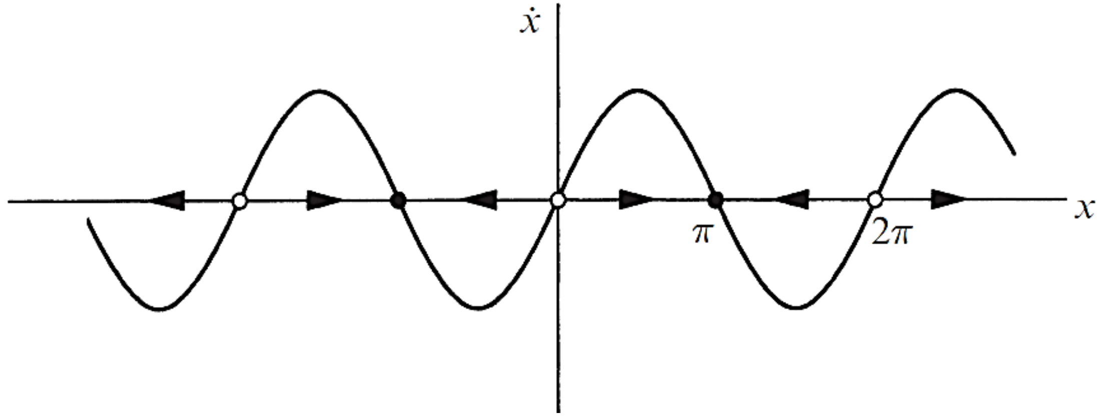
- For large -ve t's the solution will be dominated by negative eigenvalue since in these cases the exponent will be large and positive. Trajectories for large negative t's will be parallel to $\eta^{(1)}$ & moving in the same direction.

- Solutions for large positive t's will be dominated by the portion with the positive eigenvalue. Trajectories in this case will be parallel to $\eta^{(2)}$ and moving in the same direction.

Fixed points or equilibrium points

Consider the equation $\dot{x} = \sin x$

A graphical analysis of (1) is clear and simple, as shown in the figure

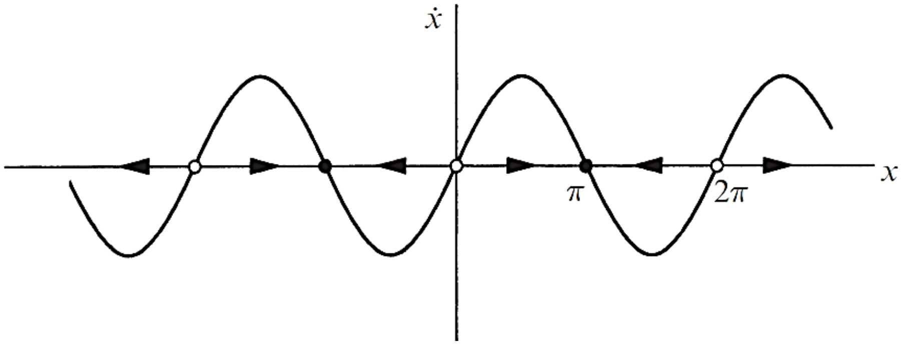


We think of *t* as time, x as the position of an imaginary particle moving along the real line, and $\dot{x}$ as the velocity of that particle.

Then the differential equation $\dot{x} = \sin x$ represents a *vector field* on the line: it dictates the velocity vector $\dot{x}$ at each *x*.

FIXED POINTS OR EQUILIBRIUM POINTS

$\dot{x} = \sin x$

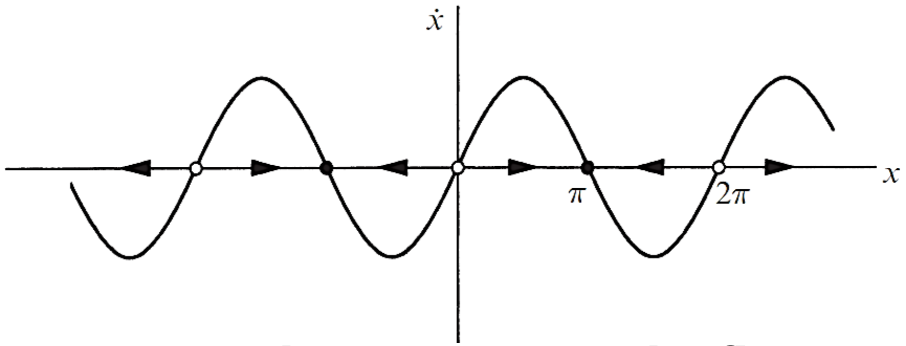


- To sketch the vector field, it is convenient to plot $\dot{x}$ versus x, and then draw arrows on the x-axis to indicate the corresponding velocity vector at each x.

- The arrows point to the right when $x>0$ and to the left when $x<0$.

A more physical way to think about the vector field: Imagine that fluid is flowing steadily along the x-axis with a velocity that varies from place to place, according to the rule

FIXED POINTS OR EQUILIBRIUM POINTS



As shown in Figure, the ***flow*** is to the right when $\dot{x} > 0$ and to the left when $\dot{x}$ <0.

At points where $\dot{x} = 0$ there is no flow such points are therefore called ***fixed points***.

Two kinds of fixed points in Figure, solid black dots represent ***stable*** fixed points (often called *attractors* or *sinks*, because the flow is toward them) and
open circles represent ***unstable*** fixed points
(also known as *repellers* or *sources*).

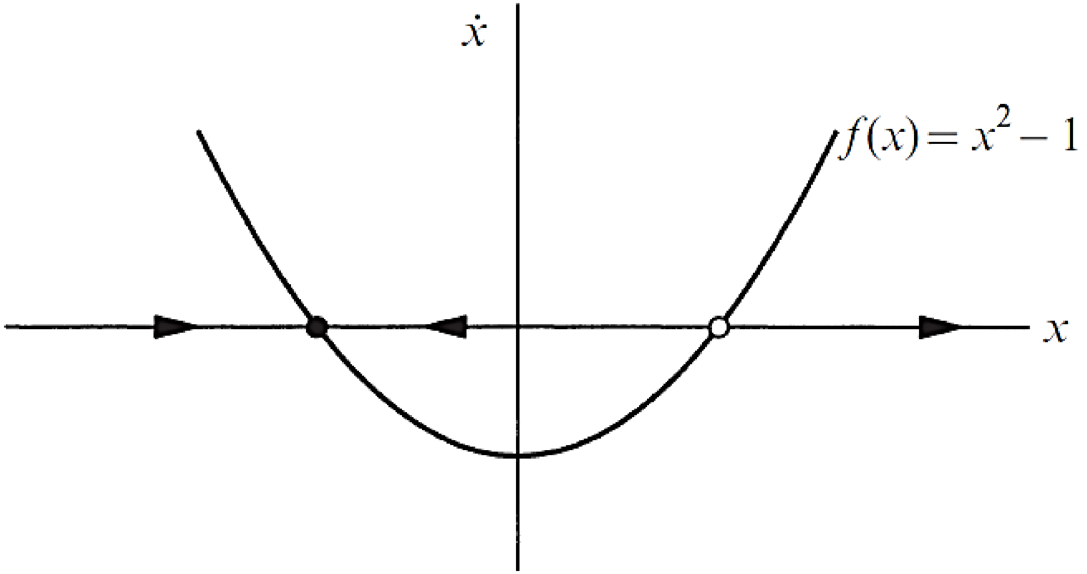
FIXED POINTS OR EQUILIBRIUM POINTS

The appearance of the phase portrait is controlled by the fixed points x^*, defined by $f(x^*) = 0$; they correspond to stagnation points of the flow.

- The solid black dot is a stable fixed point (the local flow is toward it) and the open dot is an unstable fixed point (the flow is away from it).
- In terms of the original differential equation, fixed points represent ***equilibrium*** solns (sometimes called steady, constant, or at rest solns).
- An equilibrium is defined to be stable if all sufficiently small disturbances away from it damp out in time.
- Stable equilibria are represented geometrically by stable fixed points. Unstable equilibria, in which disturbances grow in time, are represented by unstable fixed points

EXAMPLE

Solution: Here $f(x) = x^2 - 1$. To find the fixed points, we set $f(x^*) = 0$ and solve for x^*. Thus $x^* = \pm 1$. To determine stability, we plot $x^2 - 1$ and then sketch the vector field . The flow is to the right where $x^2 - 1 > 0$ and to the left where $x^2 - 1 < 0$. Thus $x^* = -1$ is stable, and $x^* = 1$ is unstable.



LINEARIZED STABILITY ANALYSIS OF NONLINEAR SYSTEMS

Consider a single-degree-of-freedom nonlinear vibratory system described by two first-order differential equations

$$\frac{dx}{dt} = f_1(x, y)$$

$$\frac{dy}{dt} = f_2(x, y)$$

where f_1 and f_2 are nonlinear functions of x and $y = \dot{x} = dx/dt$.

LINEARIZED STABILITY ANALYSIS OF NONLINEAR SYSTEMS

A study of Eqs. in the neighborhood of the singular point provides us with answers as to the stability of equilibrium. We first note that there is no loss of generality if we assume that the singular point is located at the origin (0, 0). This is because the slope $(dy)/(dx)$ of the trajectories does not vary with a translation of the coordinate axes x and y to x' and y':

$$x' = x - x_0$$

$$y' = y - y_0$$

$$\frac{dy}{dx} = \frac{dy'}{dx'}$$

LINEARIZED STABILITY ANALYSIS OF NONLINEAR SYSTEMS

If we assume $x = y = 0$ as an equilibrium point

$$f_1(0,0) = f_2(0,0) = 0$$

If f_1 and f_2 are expanded in terms of Taylor's series about the singular point (0, 0), we obtain

$$\dot{x} = f_1(x, y) = a_{11}x + a_{12}y + \text{Higher-order terms}$$

$$\dot{y} = f_2(x, y) = a_{21}x + a_{22}y + \text{Higher-order terms}$$

LINEARIZED STABILITY ANALYSIS OF NONLINEAR SYSTEMS

where

$$a_{11} = \left.\frac{\partial f_1}{\partial x}\right|_{(0,0)}, \qquad a_{12} = \left.\frac{\partial f_1}{\partial y}\right|_{(0,0)}, \qquad a_{21} = \left.\frac{\partial f_2}{\partial x}\right|_{(0,0)}, \qquad a_{22} = \left.\frac{\partial f_2}{\partial y}\right|_{(0,0)}$$

In the neighborhood of (0, 0), x and y are small; f_1 and f_2 can be approximated by linear terms only, so that Eqs. can be written as

$$\begin{Bmatrix} \dot{x} \\ \dot{y} \end{Bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{Bmatrix} x \\ y \end{Bmatrix}$$

Assume the solution of in the form

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{Bmatrix} X \\ Y \end{Bmatrix} e^{\lambda t}$$

LINEARIZED STABILITY ANALYSIS OF NONLINEAR SYSTEMS

$$\begin{bmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{bmatrix} \begin{Bmatrix} X \\ Y \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The eigenvalues λ_1 and λ_2 can be found by solving the characteristic equation

$$\begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$$

as

Eigen-values

$$\lambda_1, \lambda_2 = \frac{1}{2}\left(p \pm \sqrt{p^2 - 4q}\right)$$

where $p = a_{11} + a_{22}$ and $q = a_{11}a_{22} - a_{12}a_{21}$. If

While formulating a Jacobian based formulation: **p** becomes the trace of the determinant of the Jacobian

LINEARIZED STABILITY ANALYSIS OF NONLINEAR SYSTEMS

where C_1 and C_2 are arbitrary constants. We can note the following:

If $(p^2 - 4q) < 0$, the motion is oscillatory.

If $(p^2 - 4q) > 0$, the motion is aperiodic.

If $p > 0$, the system is unstable.

If $p < 0$, the system is stable.

If we use the transformation

$$\begin{Bmatrix} x \\ y \end{Bmatrix} = \begin{bmatrix} X_1 & X_2 \\ Y_1 & Y_2 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} = [T] \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix}$$

Matrix of eigenvectors

where $[T]$ is the modal matrix and α and β are the generalized coordinates, will be uncoupled:

$$\begin{Bmatrix} \dot{\alpha} \\ \dot{\beta} \end{Bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{Bmatrix} \alpha \\ \beta \end{Bmatrix} \quad \text{or} \quad \begin{aligned} \dot{\alpha} &= \lambda_1 \alpha \\ \dot{\beta} &= \lambda_2 \beta \end{aligned}$$

Linearized stability analysis of nonlinear systems

The solution of Eqs. can be expressed as

$$\alpha(t) = e^{\lambda_1 t}$$

$$\beta(t) = e^{\lambda_2 t}$$

Depending on the values of λ_1 and λ_2 in Eq. the singular or equilibrium points can be classified as follows

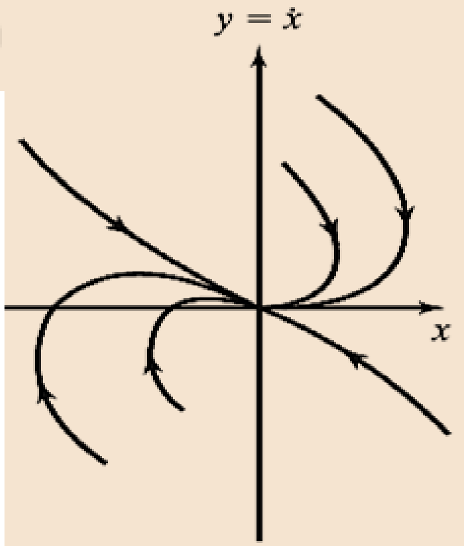
Classification of equilibrium points

Case (i)—λ_1 and λ_2 are Real and Distinct ($p^2 > 4q$).

$$\alpha(t) = \alpha_0 e^{\lambda_1 t} \qquad \text{and} \qquad \beta(t) = \beta_0 e^{\lambda_2 t}$$

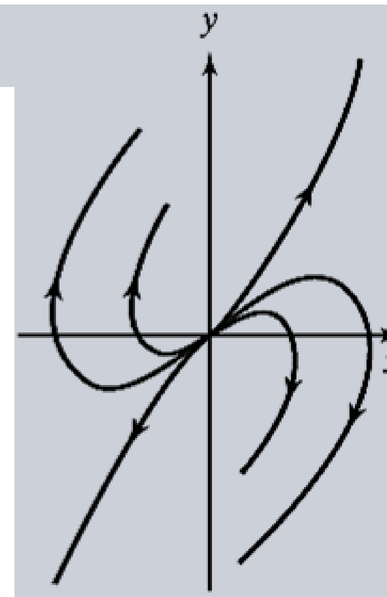
If λ_1 & λ_2 are of same sign, the type of equilibrium points are called <u>nodes or centers</u>

$\lambda_2 < \lambda_1 < 0$



Stable node

$\lambda_2 > \lambda_1 > 0$

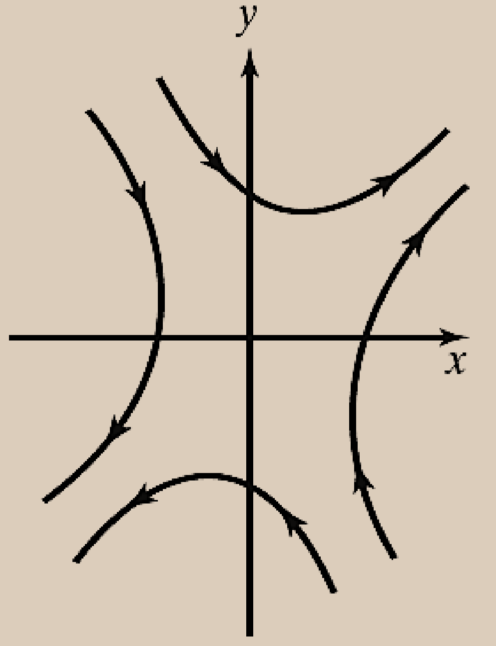


Unstable node

Classification of equilibrium points

If λ_1 and λ_2 are real but of opposite signs

The origin is called a *saddle point* and it corresponds to unstable equilibrium



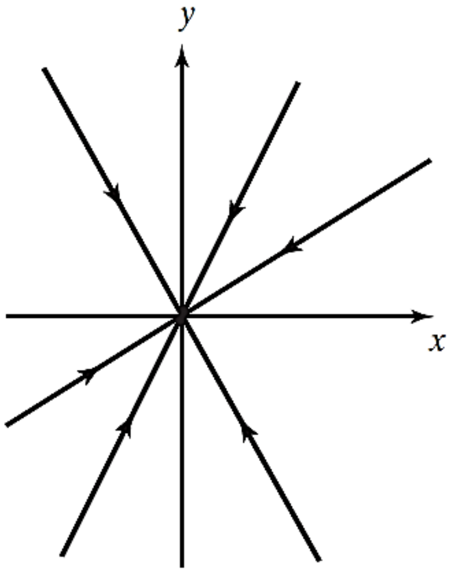
Saddle point

Classification of equilibrium points

Case (ii)—λ_1 and λ_2 are Real and Equal ($p^2 = 4q$). In this case,

$$\alpha(t) = \alpha_0 e^{\lambda_1 t} \qquad \text{and} \qquad \beta(t) = \beta_0 e^{\lambda_1 t}$$

The trajectories will be straight lines passing through the origin and the equilibrium point (origin) will be a *stable node* if $\lambda_1 < 0$ and an *unstable node* if $\lambda_1 > 0$.



Stable node

Classification of equilibrium points

Case (iii)—λ_1 and λ_2 are Complex Conjugates ($p^2 < 4q$). Let $\lambda_1 = \theta_1 + i\theta_2$ and $\lambda_2 = \theta_1 - i\theta_2$, where θ_1 and θ_2 are real. Then

$$\dot{\alpha} = (\theta_1 + i\theta_2)\alpha \qquad \text{and} \qquad \dot{\beta} = (\theta_1 - i\theta_2)\beta$$

This shows that α and β must also be complex conjugates.

$$\alpha(t) = (\alpha_0 e^{\theta_1 t})e^{i\theta_2 t}, \qquad \beta(t) = (\beta_0 e^{\theta_1 t})e^{-i\theta_2 t}$$

These equations represent logarithmic spirals

In this case the equilibrium point is called focus

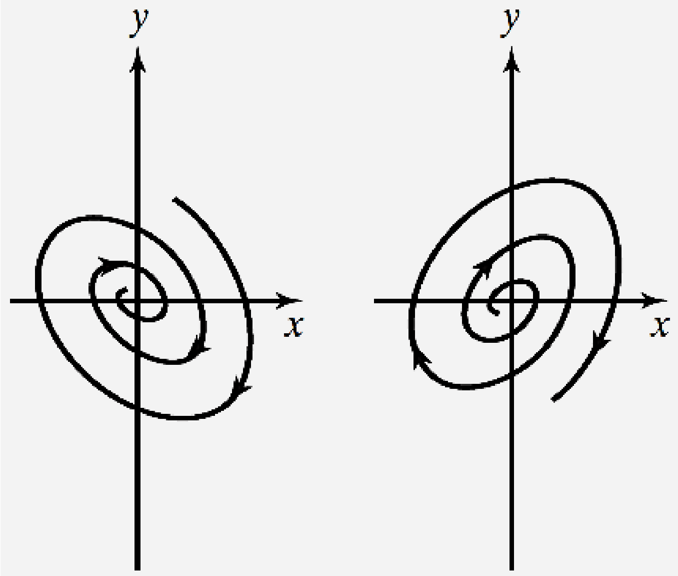
Classification of equilibrium points

$$\alpha(t) = (\alpha_0 e^{\theta_1 t}) e^{i\theta_2 t}, \qquad \beta(t) = (\beta_0 e^{\theta_1 t}) e^{-i\theta_2 t}$$

Since the factor $e^{i\theta_2 t}$ in $\alpha(t)$ represents a vector of unit magnitude rotating with angular velocity θ_2 in the complex plane, the magnitude of the complex vector $\alpha(t)$, and hence the stability of motion, is determined by $e^{\theta_1 t}$.

If $\theta_1 < 0$, the motion will be asymptotically stable and the focal point will be stable

If $\theta_1 > 0$, the focal point will be unstable



Stable focus Unstable focus

Example: PHASE PORTRAIT OF A DUFFING OSCILLATOR

Example: Phase portrait for oscillator with cubic stiffness nonlinearity (un-damped Duffing oscillator)

$$m\ddot{x} - k_1 x + k_3 x^3 = 0$$

with mass $m = 1\,\text{kg}$, negative linear stiffness $k_1 = -1\,\text{N/m}$ and cubic stiffness $k_3 = 1\,\text{N/m}^3$.

EXAMPLE: PHASE PORTRAIT OF A DUFFING OSCILLATOR

$$m\ddot{x} - k_1 x + k_3 x^3 = 0$$

First put the system into first-order form by defining $x_1 = x$ and $x_2 = \dot{x}$, such that $\ddot{x} = \dot{x}_2$. This gives

$$\dot{x}_1 = x_2 = f_1,$$
$$\dot{x}_2 = x_1 - x_1^3 = f_2.$$

Equilibrium points:

$$\boldsymbol{x}_a^* = (x_1 = 0, x_2 = 0), \boldsymbol{x}_b^* = (x_1 = 1, x_2 = 0)$$
$$\boldsymbol{x}_c^* = (x_1 = -1, x_2 = 0).$$

EXAMPLE: PHASE PORTRAIT OF A DUFFING OSCILLATOR

$$D_{\boldsymbol{x}}\mathbf{f} = \frac{\partial(f_1, f_2)}{\partial(x_1, x_2)} = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 - 3x_1^2 & 0 \end{bmatrix}.$$

For $\boldsymbol{x}_a^* = (x_1 = 0, x_2 = 0)$, the Jacobian becomes

$$D_{\boldsymbol{x}_a^*}\mathbf{f} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

So for equilibrium point $\boldsymbol{x}_a^*$, $\mathrm{tr}(A) = 0$ and $\det(A) = -1$

SADDLE POINT

EXAMPLE: PHASE PORTRAIT OF A DUFFING OSCILLATOR

For equilibrium point $\boldsymbol{x}_b^* = (x_1 = 1, x_2 = 0)$, the Jacobian becomes

$$D_{\boldsymbol{x}_a^*}\mathbf{f} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix},$$

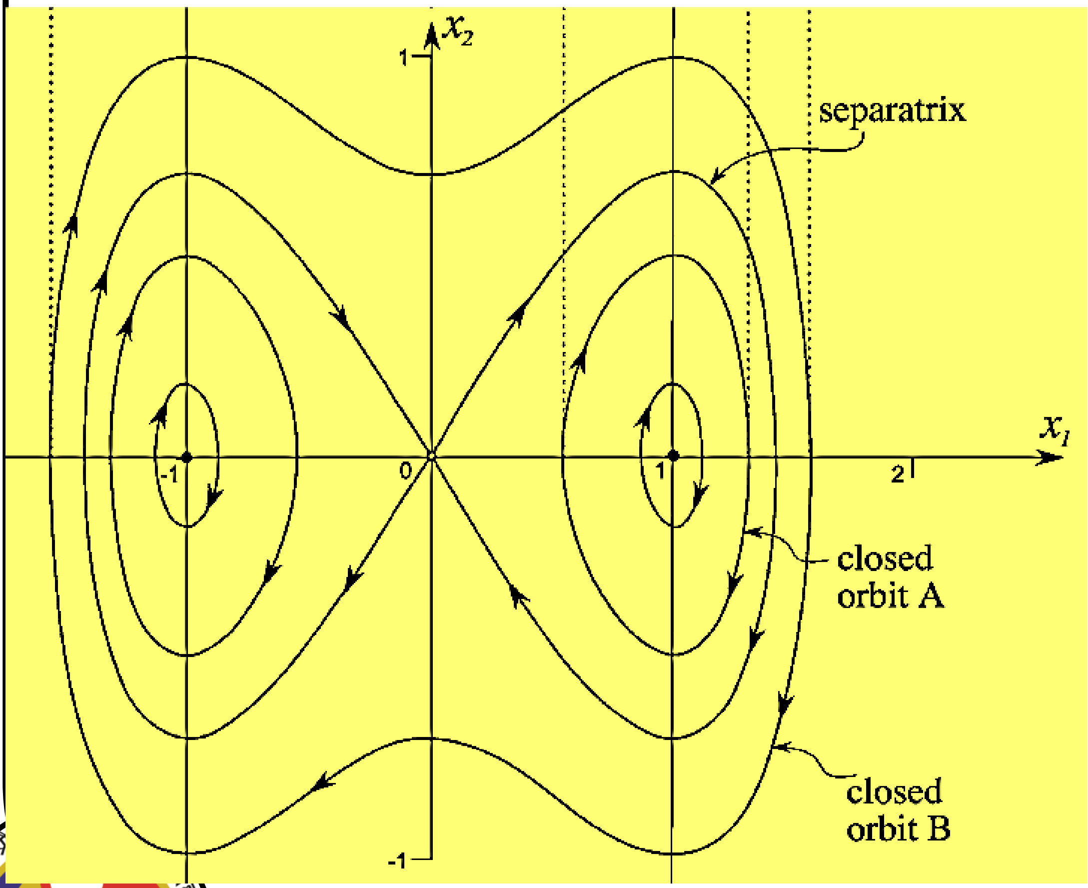
so in this case $\mathrm{tr}(A) = 0$ and $\det(A) = 2$

CENTRE

Equilibrium point $\boldsymbol{x}_c^* = (x_1 = -1, x_2 = 0)$ has the same Jacobian as equilibrium point $\boldsymbol{x}_b^*$

CENTRE

EXAMPLE: PHASE PORTRAIT OF A DUFFING OSCILLATOR



Here the separatrix marks the boundary between

(i) the orbits confined around each of the centre equilibrium points

(ii) orbits which enclose both

A further analogy is to imagine the phase space orbits as contours. These contours indicate lines of constant energy

State Space and Mechanical Energy

Consider an unforced undamped linear oscillator

$$m\ddot{x} + kx = 0$$

Consideringthe work done over a small increment of distance dx, as the mass moves from resting x = 0 to an arbitrary x value gives the integral

$$\int_0^x (m\ddot{x} + kx)\mathrm{d}x = m\int_0^x \ddot{x}\mathrm{d}x + k\int_0^x x\mathrm{d}x = E_t = \frac{1}{2}mv^2 + \frac{1}{2}kx^2$$

State Space and Mechanical Energy

Now a direct link can be made between the system state space and the energy in the system. To see this, first notice that in terms of state variables the velocity, $v = \dot{x} = x_2$ and the displacement $x = x_1$. Now consider the unforced, undamped nonlinear oscillator

$$m\ddot{x} + p(x) = 0 \rightsquigarrow mv\frac{\mathrm{d}v}{\mathrm{d}x} + p(x) = 0,$$

where $p(x)$ is the stiffness function. Integrating to find the energy gives

$$\frac{1}{2}mv^2 + \int_0^x p(x) = E_t \rightsquigarrow \frac{1}{2}mv^2 + V(x) = E_t,$$

where $V(x) = \int_0^x p(x)$ is called the *potential* function.

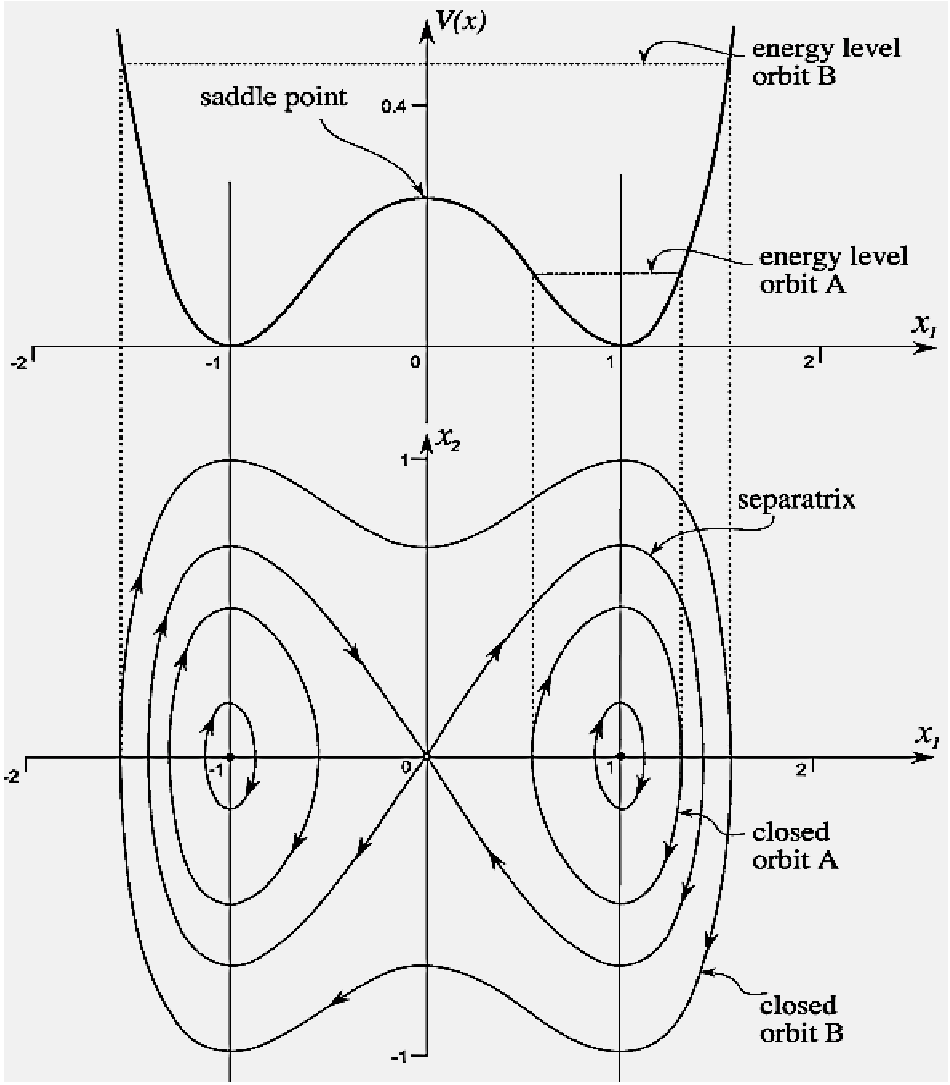
EXAMPLE: POTENTIAL FUNCTION OF A DUFFING OSCILLATOR

The potential function can be found by integrating $p(x_1) = -x_1 + x_1^3$

$$V(x_1) = -\frac{1}{2}x_1^2 + \frac{1}{3}x_1^3 + \frac{1}{4}$$

where the $\frac{1}{4}$ constant ensures that the potential function is always positive

EXAMPLE: POTENTIAL FUNCTION OF A DUFFING OSCILLATOR



EXAMPLE: POTENTIAL FUNCTION OF A DUFFING OSCILLATOR

The system plotted in Fig actually has a negative linear stiffness $k_1 = -1$ which explains why there is a saddle point at the origin.

This type of system may at first seem to have limited physical applications, but it can be used to model an interesting class of systems which have bi-stability.

Or, in other words, they have two stable configurations (like the two equilibrium points at +1 and -1 respectively

Example: potential function of a Duffing oscillator

The form of *V(x)* shown in Fig is often called a double potential well.

The sides of the well continue to extend upwards, and energy levels for two different orbits are shown in Fig.

Orbit A is inside the potential well around the equilibrium point at $x_1 = 1, x_2 = 0$.
Orbit B has a much higher energy level and is not confined to either of the centre equilibrium points.

Here the separatrix marks the boundary between
(i) the orbits confined to the potential wells around each of the centre equilibrium points and
(ii) orbits which enclose both.

LIMIT CYCLES

In certain vibration problems involving nonlinear damping, the trajectories, starting either very close to the origin, or, far away from the origin, tend to a single closed curve, which corresponds to a periodic solution of the system.

An interesting property of the solution is that all the trajectories, irrespective of the initial conditions, approach asymptotically a particular closed curve, known as the *limit cycle*, which represents a steady-state periodic (but not harmonic)
oscillation.
This is a phenomenon that can be observed only with certain nonlinear vibration problems and not in any linear problem.

LIMIT CYCLES: *VAN DER POL EQUATION*

$$\ddot{x} - \alpha(1 - x^2)\dot{x} + x = 0, \qquad \alpha > 0$$

- This equation exhibits, the essential features of some vibratory systems, such as certain electrical feedback circuits controlled by valves where there is a source of power that increases with the amplitude of vibration.

- Van der Pol invented it by introducing a type of damping that is negative for small amplitudes but becomes positive for large amplitudes

- In this equation, he assumed the damping term to be a multiple of $-(1 - x^2)\,\dot{x}$ in order to make the magnitude of the damping term independent of the sign of x.

INTRODUCTION TO BIFURCATION THEORY

- As we've seen in Stability Analysis, the dynamics of vector fields on the line or 2D-plane is very limited: all solutions either settle down to equilibrium or head out to $\pm\infty$.
- What's more interesting is *Dependence on parameters*.
- The qualitative structure of the flow can change as parameters are varied.
- In particular, fixed points can be created or destroyed, or their stability can change.

- These qualitative changes in the dynamics are called ***bifurcations***, and the parameter values at which they occur are called ***bifurcation points.***

Saddle-node bifurcation

Bifurcations provide models of transitions and instabilities as some *control parameter* is varied.

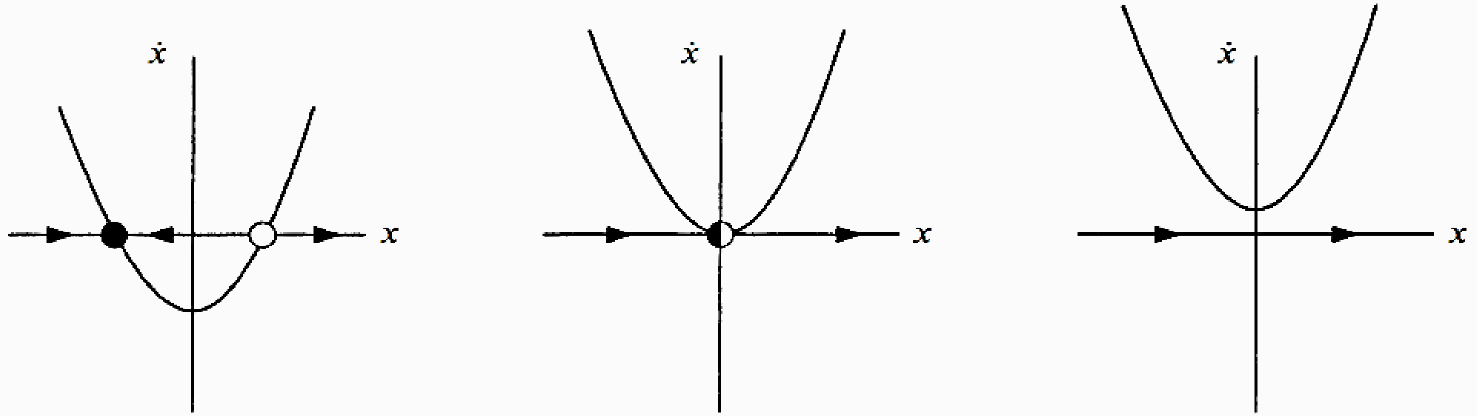
Saddle-Node Bifurcation

The saddle-node bifurcation is the basic mechanism by which fixed points are *created and destroyed*. As a parameter is varied, two fixed points move toward each other, collide, and mutually annihilate.

SADDLE-NODE BIFURCATION

The prototypical example of a saddle-node bifurcation is given by the first-order system $\dot{x} = r + x^2$

where r is a parameter, which may be positive, negative, or zero. When r is negative, there are two fixed points, one stable and one unstable



(a) $r<0$ (b) $r=0$ (c) $r>0$

SADDLE-NODE BIFURCATION

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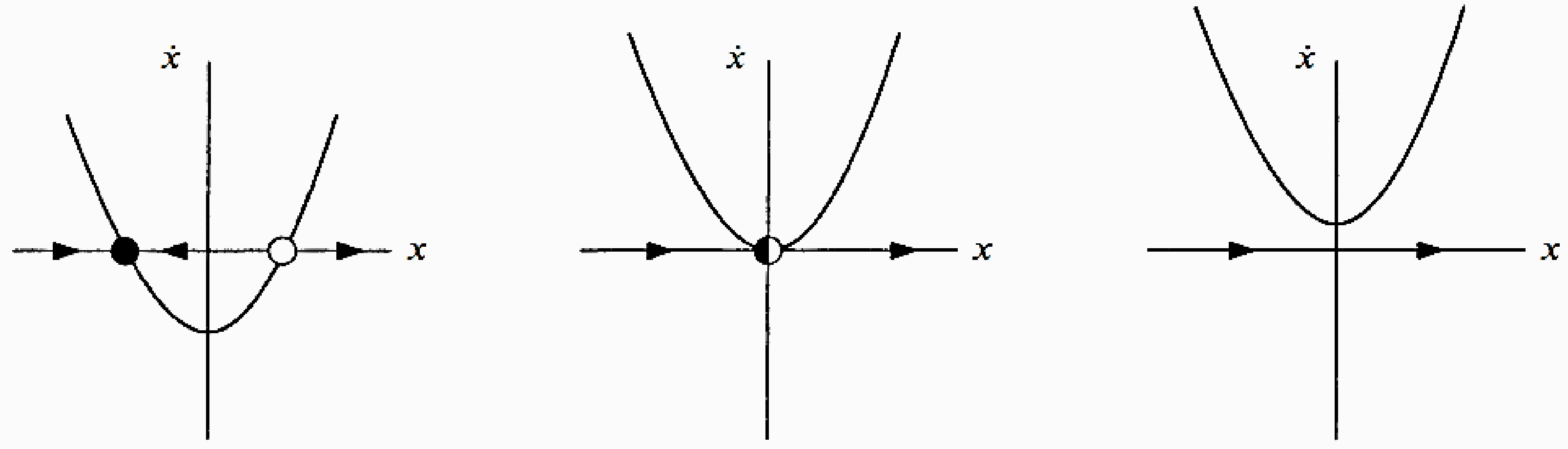
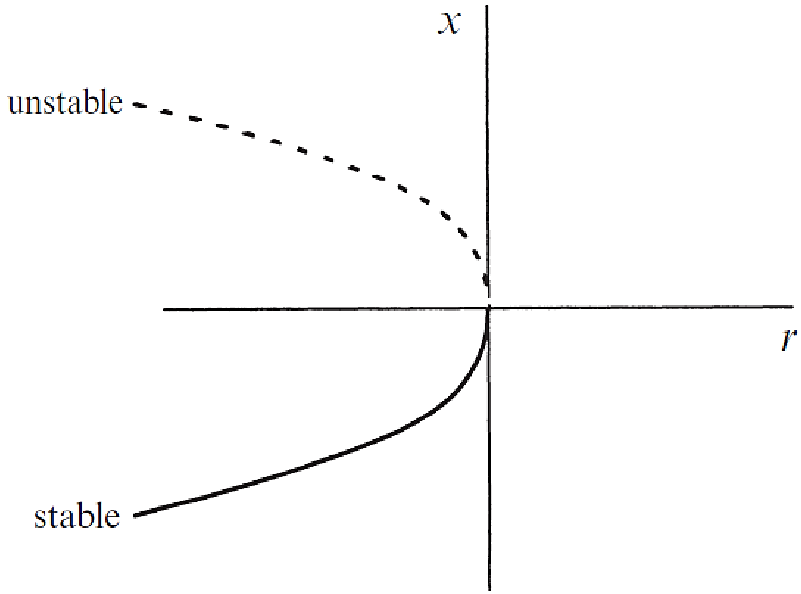


(a) $r<0$ (b) $r=0$ (c) $r>0$

- As r approaches 0 from below, the parabola moves up and the two fixed points move toward each other. When $r = 0$, the fixed points coalesce into a half-stable fixed point at $x^* = 0$
- This type of fixed point is extremely delicate: vanishes as soon as $r > 0$, and now there are no fixed points at all

SADDLE-NODE BIFURCATION: DIAGRAMS



This picture is called the ***bifurcation diagram*** for the saddle-node bifurcation

SADDLE-NODE BIFURCATION

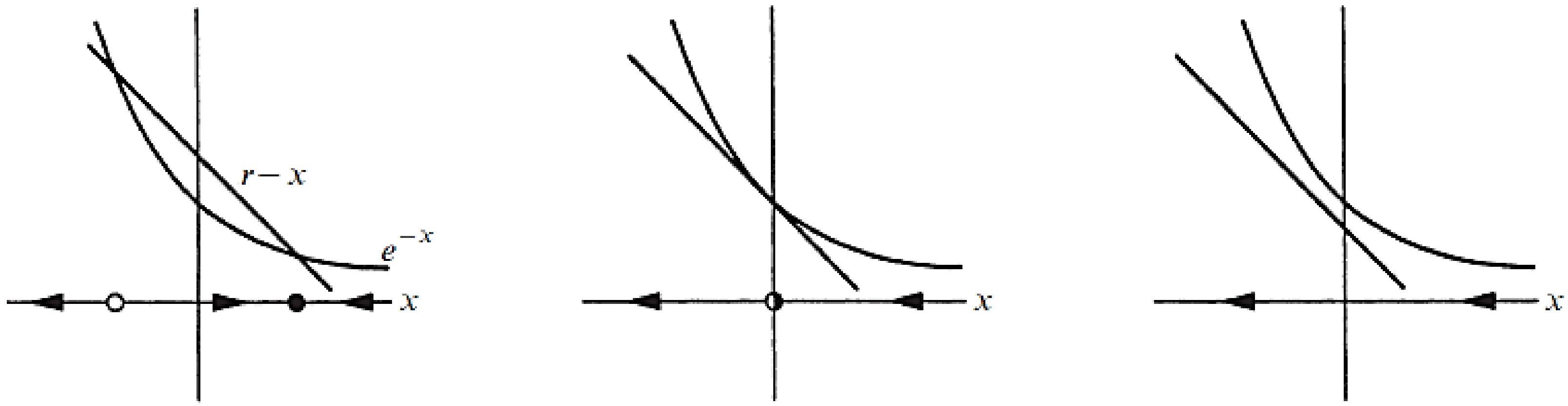
Show that the first-order system $\dot{x} = r - x - e^{-x}$ undergoes a saddle-node bifurcation as r is varied, and find the value of r at the bifurcation point.

Difficulty: We can't find the fixed points explicitly as a function of r by setting *f(x)=0*;

The point is that the two functions $r - x$ and e^{-x} have much more familiar graphs than their difference $r - x - e^{-x}$.

Plot $(r - x)$ and e^{-x} in the same figure

SADDLE-NODE BIFURCATION



- Thus, intersections of the line and the curve correspond to fixed points for the system
- This picture also allows us to read off the direction of flow on the *x*-axis: the flow is to the right where the line lies above the curve

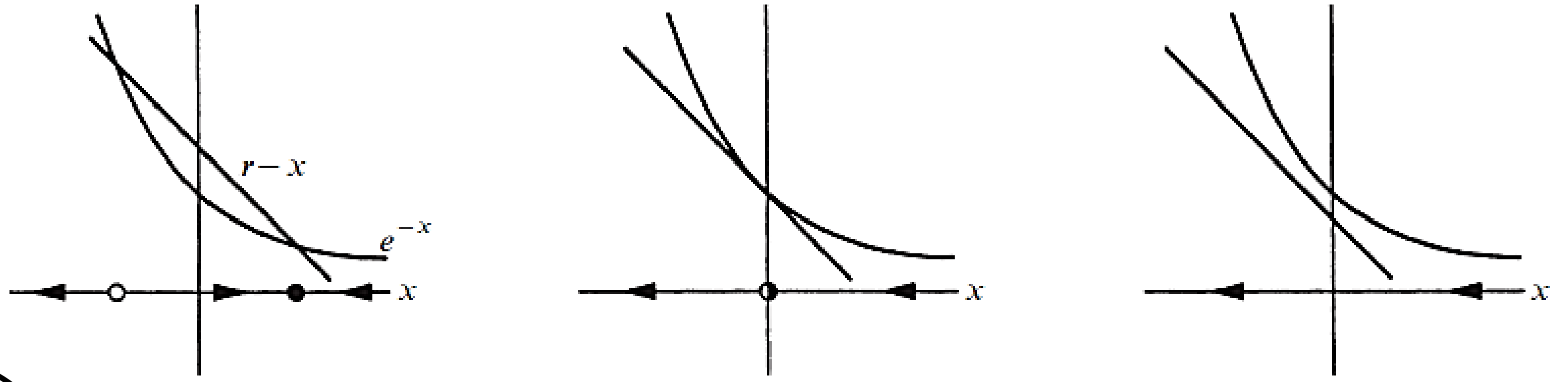
$(r - x) > e^{-x}$ in therefore $\dot{x} > 0$

Hence, the fixed point on the right is stable, and the one on the left is unstable

SADDLE-NODE BIFURCATION

Now, start decreasing the parameter *r*. The line *r* -*x* slides down and the fixed points approach each other. At some critical value $r = r_c$, the line becomes *tangent* to the curve and the fixed points coalesce in a saddle-node bifurcation

For *r* below this critical value, the line lies below the curve and there are no fixed points



Saddle-node bifurcation

To find the bifurcation point r_c, we impose the condition that the graphs of $r - x$ and e^{-x} intersect *tangentially*. Thus we demand equality of the functions *and* their derivatives:

$$e^{-x} = r - x$$

and

$$\frac{d}{dx}e^{-x} = \frac{d}{dx}(r - x).$$

The second equation implies $-e^{-x} = -1$, so $x = 0$. Then the first equation yields $r = 1$. Hence the bifurcation point is $r_c = 1$, and the bifurcation occurs at $x = 0$. ■

PITCHFORK BIFURCATION

- This bifurcation is common in physical problems that have a ***symmetry***.

  For example, many problems have a spatial symmetry between left & right.

- In such cases, fixed points tend to appear and disappear in symmetrical pairs. In the buckling of column, the column is stable in the vertical position if the load is small.

- In this case there is a stable fixed point corresponding to zero deflection. But if the load exceeds the buckling threshold, the beam may buckle to *either* the left or the right.

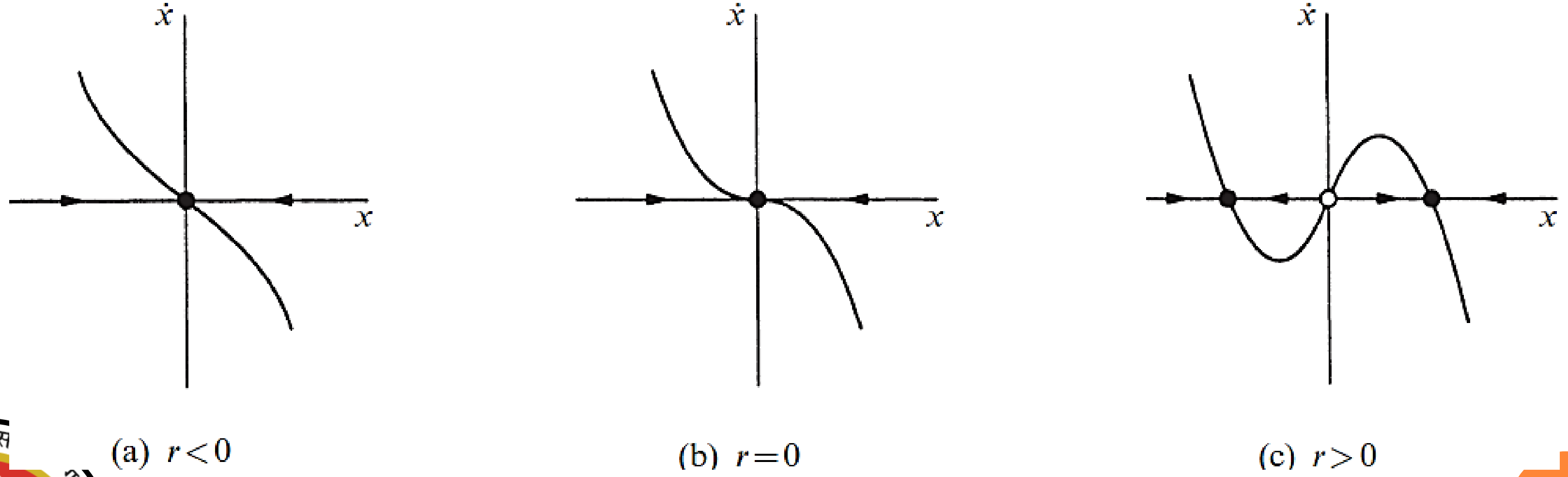
- The vertical position has gone unstable, and two new symmetrical fixed points, corresponding to left- and right-buckled configurations, have been born.

Pitchfork Bifurcation

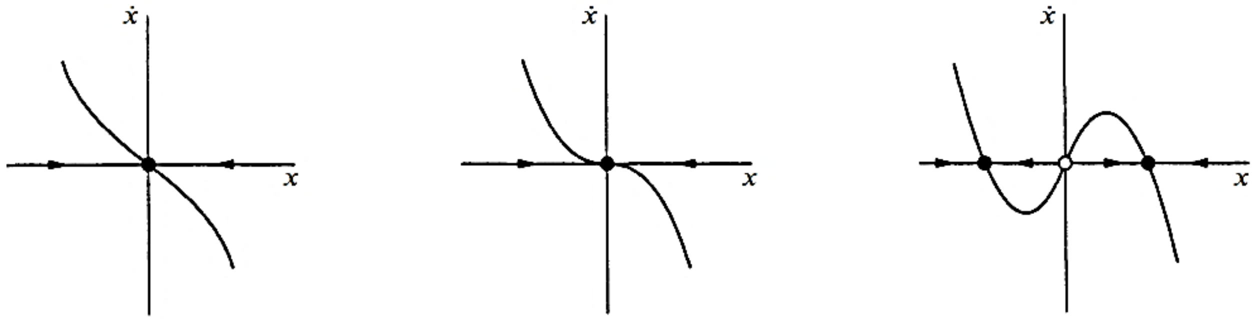
$\dot{x} = rx - x^3$

Observe that if x is replaced by $-x$, nothing changes : Symmetry

Vector fields:



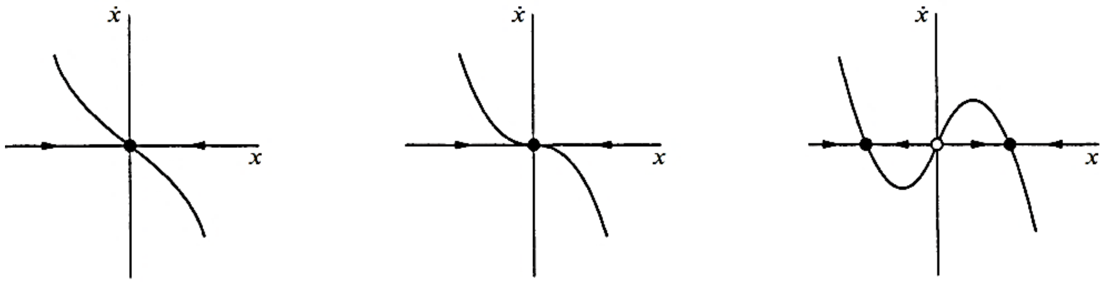
PITCHFORK BIFURCATION



(a) $r<0$ (b) $r=0$ (c) $r>0$

When $r < 0$, the origin is the only fixed point, and it is stable. When $r = 0$, the origin is still stable, but much more weakly so, since the linearization vanishes. Now solutions no longer decay exponentially fast—instead the decay is a much slower algebraic function of time

PITCHFORK BIFURCATION



(a) $r<0$ (b) $r=0$ (c) $r>0$

Finally, when $r > 0$, the origin has become unstable. Two new stable fixed points appear on either side of the origin, symmetrically located at $x^* = \pm\sqrt{r}$.