CE 513: Statistical Methods in CE

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ITO CALCULUS-1

Ito Integrals

$$I_n(W) = \sum_{i=1}^{n} W(s_{i-1})\{W(s_i) - W(s_{i-1})\}$$

- $W(t)$ *is a standard Wiener process with* $\sigma^2 = 1, \mu = 0$
- Then the Ito sum converges to Ito integral in mean-square

$$I_n(W) \xrightarrow{2} \int_0^t W(s)\, dW(s)$$

Ito Integral

$$I_n(W) = \int_0^t W(s)\, dW(s) = \frac{W^2(t)}{2} - \frac{t}{2}$$

The above proof requires the development of quadratic variation theorem which will be studied next

Moments:

$E[\int_0^t W(s)\, dW(s)] = 0$

$$Var[\int_0^t W(s)\, dW(s)] = Autocov[\int_0^t W(s)\, dW(s)] = t^2/2$$

Quadratic variation of Wiener Processes

Again, the considerations are based on an adequate partition of the interval $[0, t]$,

$$P_n([0, t]) : 0 = s_0 < s_1 < \ldots < s_n = t .$$

For a function g the **variation** over this partition is defined as :

$$V_n(g, t) = \sum_{i=1}^{n} |g(s_i) - g(s_{i-1})| .$$

If the limit exists independently of the decomposition for $n \to \infty$ one says that g is of finite variation and writes :

$$V_n(g, t) \to V(g, t) , \quad n \to \infty .$$

Quadratic variation of Wiener Processes

The finite sum $V_n(g,t)$ measures for a certain partition the absolute increments of the function g on the interval $[0,t]$. If the function evolves sufficiently smooth, then $V(g,t)$ takes on a finite value for $(n \to \infty)$.

For very jagged functions

an increasing refinement $(n \to \infty)$ the increments of the graph of g become larger and larger even for fixed t, such that g is not of finite variation.

Quadratic variation of Wiener Processes

PROPOSITION-1 **(Variation of Continuously Differentiable Functions)** *Let g be a continuously differentiable function with derivative g' on $[0, t]$. Then g is of finite variation and it holds that*

$$V(g, t) = \int_0^t |g'(s)| \, ds.$$

Example: Let's see when this fails

(Sine Wave) Let us consider a sine cycle of the frequency k on the interval $[0, 2\pi]$:

$$g_k(s) = \sin(ks), \quad k = 1, 2, \ldots .$$

Quadratic variation of Wiener Processes

(Sine Wave) Let us consider a sine cycle of the frequency k on the interval $[0, 2\pi]$:

$$g_k(s) = \sin(ks), \quad k = 1, 2, \ldots .$$

$$g_k'(s) = k \cos(ks).$$

Accounting for the sign one obtains as the variation:

$$\begin{aligned} V(g_1, 2\pi) &= \int_0^{2\pi} |\cos(s)| \, ds = 4 \int_0^{\pi/2} \cos(s) \, ds \\ &= 4 \left(\sin\left(\frac{\pi}{2}\right) - \sin(0) \right) \\ &= 4 , \end{aligned}$$

$$\begin{aligned} V(g_2, 2\pi) &= \int_0^{2\pi} 2 \, |\cos(2s)| \, ds = 8 \int_0^{\pi/4} 2 \cos(2s) \, ds \\ &= 8 \left(\sin\left(\frac{\pi}{2}\right) - \sin(0) \right) \\ &= 8 , \end{aligned}$$

Quadratic variation of Wiener Processes

$$V(g_k, 2\pi) = \int_0^{2\pi} k\,|\cos(ks)|\;ds = 4k\int_0^{\pi/2k} k\cos(ks)\,ds$$
$$= 4k\left(\sin\left(\frac{\pi}{2}\right) - \sin(0)\right)$$
$$= 4k\,.$$

It can be observed, how the sum of (absolute) differences in amplitude grows with k growing. Accordingly, the absolute variation of $g_k(s) = \sin(ks)$ multiplies with k.

Consequently the absolute variation is not finite for $g_k = \sin ks$ in the limiting case $k \to \infty$

Quadratic variation of Wiener Processes

Quadratic Variation

In the same way as $V_n(g,t)$ a q-variation can be defined where we are only interested in the case $q = 2$, – the quadratic variation:

$$Q_n(g,t) = \sum_{i=1}^{n} |g(s_i) - g(s_{i-1})|^2 = \sum_{i=1}^{n} (g(s_i) - g(s_{i-1}))^2 \ .$$

As would seem natural, g is called of finite quadratic variation if it holds that

$$Q_n(g,t) \ \rightarrow \ Q(g,t)\,, \quad n \rightarrow \infty\,.$$

Quadratic variation of Wiener Processes

PROPOSITION-2 **(Absolute and Quadratic Variation)** *Let g be a continuous function on $[0, t]$. It then holds for $n \to \infty$:*

$$V_n(g, t) \;\to\; V(g, t) < \infty$$

implies

$$Q_n(g, t) \;\to\; 0\,.$$

If g is a stochastic process, then "$\to$" is to be understood as convergence in mean square.

Quadratic variation of Wiener Processes

(Quadratic Variation of the WP) *For the Wiener process with* $n \to \infty$

$$Q_n(W,t) \stackrel{2}{\to} t = Q(W,t)\,.$$

The expression $Q(W,t) = t$ characterizes the level of jaggedness or irregularity of the Wiener process on the interval $[0,t]$.

Rephrasing it one can state

If the Wiener process was continuously differentiable, then it would be of finite variation due to Proposition-**1** and it would have a vanishing quadratic variation due to Proposition-**2** . However, this is just not the case.

Quadratic variation of Wiener Processes

To prove: $Q_n(W,t) \to t = Q(W,t)$

1st stage: Show $E\{[Q_n(W,t) - t]^2\} \to 0$ as $\lim n \to \infty$

$$Q_n(W,t) = \sum_{i=1}^{n} (W(s_i) - W(s_{i-1}))^2$$

$$\mathrm{E}\,(Q_n(W,t)) = \sum_{i=1}^{n} \mathrm{Var}\,(W(s_i) - W(s_{i-1}))$$

$$= \sum_{i=1}^{n} (s_i - s_{i-1}) = s_n - s_0 = t$$

Quadratic variation of Wiener Processes

To prove: $Q_n(W,t) \to t$

2nd stage: Show $Var\,\{Q_n(W,t)\} \to 0$ as $\lim n \to \infty$

$$\mathrm{Var}\,(Q_n(W,t)) = \sum_{i=1}^{n} \mathrm{Var}\left[(W(s_i) - W(s_{i-1}))^2\right]$$

$$\begin{aligned}\mathrm{Var}\left[(W(s_i) - W(s_{i-1}))^2\right] &= \mathrm{E}\left[(W(s_i) - W(s_{i-1}))^4\right] - \left(\mathrm{E}\left[(W(s_i) - W(s_{i-1}))^2\right]\right)^2 \\ &= 3\,[\mathrm{Var}(W(s_i) - W(s_{i-1}))]^2 - (s_i - s_{i-1})^2 \\ &= 2\,(s_i - s_{i-1})^2\,.\end{aligned}$$

$$\begin{aligned}\mathrm{Var}(Q_n(W,t)) &= 2\sum_{i=1}^{n}(s_i - s_{i-1})^2 \\ &\leq 2 \max_{1\leq i\leq n}(s_i - s_{i-1}) \sum_{i=1}^{n}(s_i - s_{i-1}) \\ &= 2 \max_{1\leq i\leq n}(s_i - s_{i-1})\,(s_n - s_0) \\ &\to 0\,, \quad n \to \infty\,,\end{aligned}$$

General Ito Integrals

$$I_n(W) = \sum_{i=1}^{n} X(s_{i-1})\{W(s_i) - W(s_{i-1})\}$$

• If X (t) is a process with finite variance where the variance varies continuously in with time,

•If X (t) *only depends on the past of the WP, W (s) with s<=t,*

Then the Ito sum converges uniquely and independently of the partition.

$$I_n(W) \to \int_0^t X(s)\, dW(s)$$

Moments:

E $[\int_0^t X(s)\, dW(s)] = 0$ Var $[\int_0^t X(s)\, dW(s)] = \int_0^t X^2(s)\, dW(s)$

Diffusions

• For modeling Stochastic-Differential Equations, Ito integral is an important ingredient.

• However, it gains its true importance only when combined with Riemann integrals.

• In general the sum of both integrals constitutes so-called **diffusions**

Hence, we now define processes $X(t)$ (with starting value $X(0)$)

$$X(t) = X(0) + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dW(s)\,.$$

Frequently, we will write this integral equation in differential form :

$$dX(t) = \mu(t)\, dt + \sigma(t)\, dW(t)\,.$$

Diffusions

Rectangular Snip

$$X(t) = X(0) + \int_0^t \mu(s)\, ds + \int_0^t \sigma(s)\, dW(s)\,.$$

Frequently, we will write this integral equation in differential form

$$dX(t) = \mu(t)\, dt + \sigma(t)\, dW(t)\,.$$

In general, $\mu(s)$ and $\sigma(s)$ are stochastic; particularly, they are allowed to be dependent on $X(s)$ itself. Therefore, we write $\mu(s)$ and $\sigma(s)$ as abbreviations for functions which firstly explicitly depend on time and secondly depend on X simultaneously:

$$\mu(s) = \mu\,(s, X(s))\ , \quad \sigma(s) = \sigma\,(s, X(s))$$

Processes μ and σ satisfying these conditions are used to define diffusions $X(t)$:

$$dX(t) = \mu\,(t, X(t))\ dt + \sigma\,(t, X(t))\ dW(t)\,, \quad t \in [0, T]\,.$$

Diffusions

Example *(Brownian Motion with Drift)* We consider the Brownian motion with drift and a starting value 0:

$$X(t) = \mu t + \sigma W(t) = \mu \int_0^t ds + \sigma \int_0^t dW(s) .$$

Therefore, the differential notation reads

$$dX(t) = \mu\, dt + \sigma\, dW(t) .$$

Hence, this is a diffusion whose drift and volatility are constant:

$$\mu(t, X(t)) = \mu \quad \text{and} \quad \sigma(t, X(t)) = \sigma .$$

Propositions

Proposition **(Symbolic Notation)** *It holds for $n \to \infty$*

$$(a)\ \sum_{i=1}^{n} (W(s_i) - W(s_{i-1}))^2 \xrightarrow{2} \int_0^t (dW(s))^2 = t\,,$$

$$(b)\ \sum_{i=1}^{n} (W(s_i) - W(s_{i-1}))\ (s_i - s_{i-1}) \xrightarrow{2} \int_0^t dW(s)\,ds = 0$$

$$(c)\ \sum_{i=1}^{n} (s_i - s_{i-1})^2 \to \int_0^t (ds)^2 = 0\,.$$

Example

Prove:

$$\sum_{i=1}^{n} (W(s_i) - W(s_{i-1}))\ (s_i - s_{i-1}) \xrightarrow{2} \int_0^t dW(s)\, ds = 0$$

The LHS of the above expression is also called Quadratic Covariation CV_n

$$CV_n = \sum_{i=1}^{n} (W(s_i) - W(s_{i-1}))\ (s_i - s_{i-1})$$

The above proof tantamounts to proving $\mathrm{MSE}(CV_n, 0) \to 0$.

It is easy to see that E $(CV_n) = 0$

Hence, it remains to be shown that this variance tends to zero: Due to the independence of the increments of the WP, one determines

$$\mathrm{Var}(CV_n) = \sum_{i=1}^{n} \mathrm{Var}\,(W(s_i) - W(s_{i-1}))\ (s_i - s_{i-1})^2\,,$$

Example

and hence

$$\begin{aligned}\mathrm{Var}(CV_n) &= \sum_{i=1}^{n}(s_i - s_{i-1})^3 \\ &\leq \max_{1\leq i\leq n}(s_i - s_{i-1}) \sum_{i=1}^{n}(s_i - s_{i-1})^2 \\ &= \max_{1\leq i\leq n}(s_i - s_{i-1})\, Q_n(id, t) \\ &\to 0\,,\end{aligned}$$

ITO's Lemma for Diffusions

Proposition 1 **(Ito's Lemma with One Dependent Variable)** *Let $g : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable and $X(t)$ a diffusion on $[0, T]$*

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t)\,.$$

Then it holds that

$$dg\,(X(t)) = g'\,(X(t))\;dX(t) + \frac{1}{2}\,g''\,(X(t))\;\sigma^2(t)\,dt\,.$$

If $X(t) = W(t)$ is a Wiener process, i.e. $\mu(t) = 0$ and $\sigma(t) = 1$, then Corollary 1 is obtained as a special case.

The statement in Proposition 1 is given somewhat succinctly It can be condensed even more by suppressing the dependence on time:

$$dg\,(X) = g'\,(X)\;dX + \frac{1}{2}\,g''\,(X)\;\sigma^2\,dt\,.$$

ITO’s Lemma

Corollary 1 (Ito’s Lemma for WP) *Let $g : \mathbb{R} \to \mathbb{R}$ be twice continuously differentiable. Then it holds that*

$$dg\,(W(t)) = g'\,(W(t))\;dW(t) + \frac{1}{2}\,g''\,(W(t))\;dt\,.$$

In integral form this corollary to Ito’s lemma is to be read as follows:

$$g\,(W(t)) = g\,(W(0)) + \int_0^t g'\,(W(s))\;dW(s) + \frac{1}{2}\int_0^t g''\,(W(s))\;ds\,.$$

Strictly speaking, this integral equation is the statement of the corollary, which is abbreviated by the differential notation. However, in doing so it must not be forgotten that the WP is not differentiable. Sometimes one also writes even more briefly:

$$dg\,(W) = g'(W)\,dW + \frac{1}{2}\,g''(W)\,dt\,.$$

Example

Example (Differential of the Exponential Function) Let a diffusion $X(t)$ be given,

$$dX(t) = \mu(t)\,dt + \sigma(t)\,dW(t)\,.$$

Then, how does the differential of $e^{X(t)}$ read? This example is particularly easy to calculate as it holds for $g(x) = e^x$ that:

$$g''(x) = g'(x) = g(x) = e^x.$$

Hence Ito's lemma yields:

$$\begin{aligned} de^{X(t)} &= e^{X(t)} dX(t) + \frac{e^{X(t)}}{2}\sigma^2(t)dt \\ &= e^{X(t)}\left(\mu(t) + \frac{\sigma^2(t)}{2}\right)dt + e^{X(t)}\,\sigma(t)dW(t). \end{aligned}$$

If $X(t)$ is deterministic, i.e. $\sigma(t) = 0$, then it results

$$\frac{de^{X(t)}}{dt} = e^{X(t)}\frac{dX(t)}{dt},$$

which just corresponds to the traditional chain rule (outer derivative times inner derivative).