CE 513: Statistical Methods in Civil Engineering

Random process

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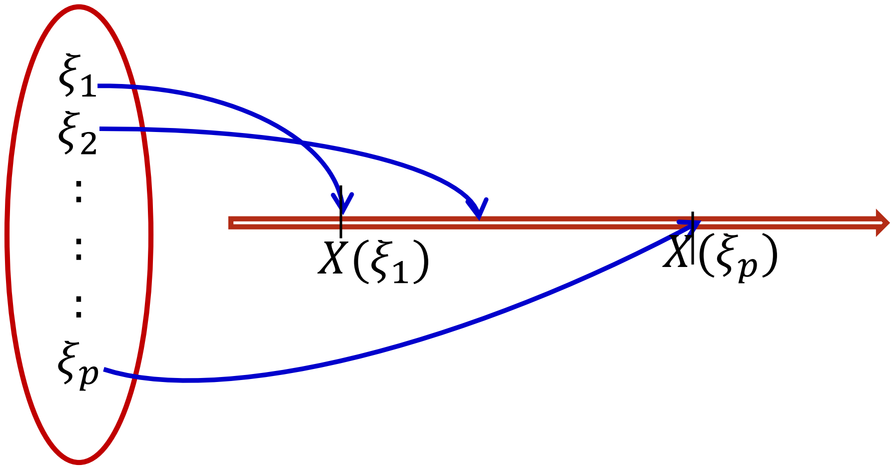
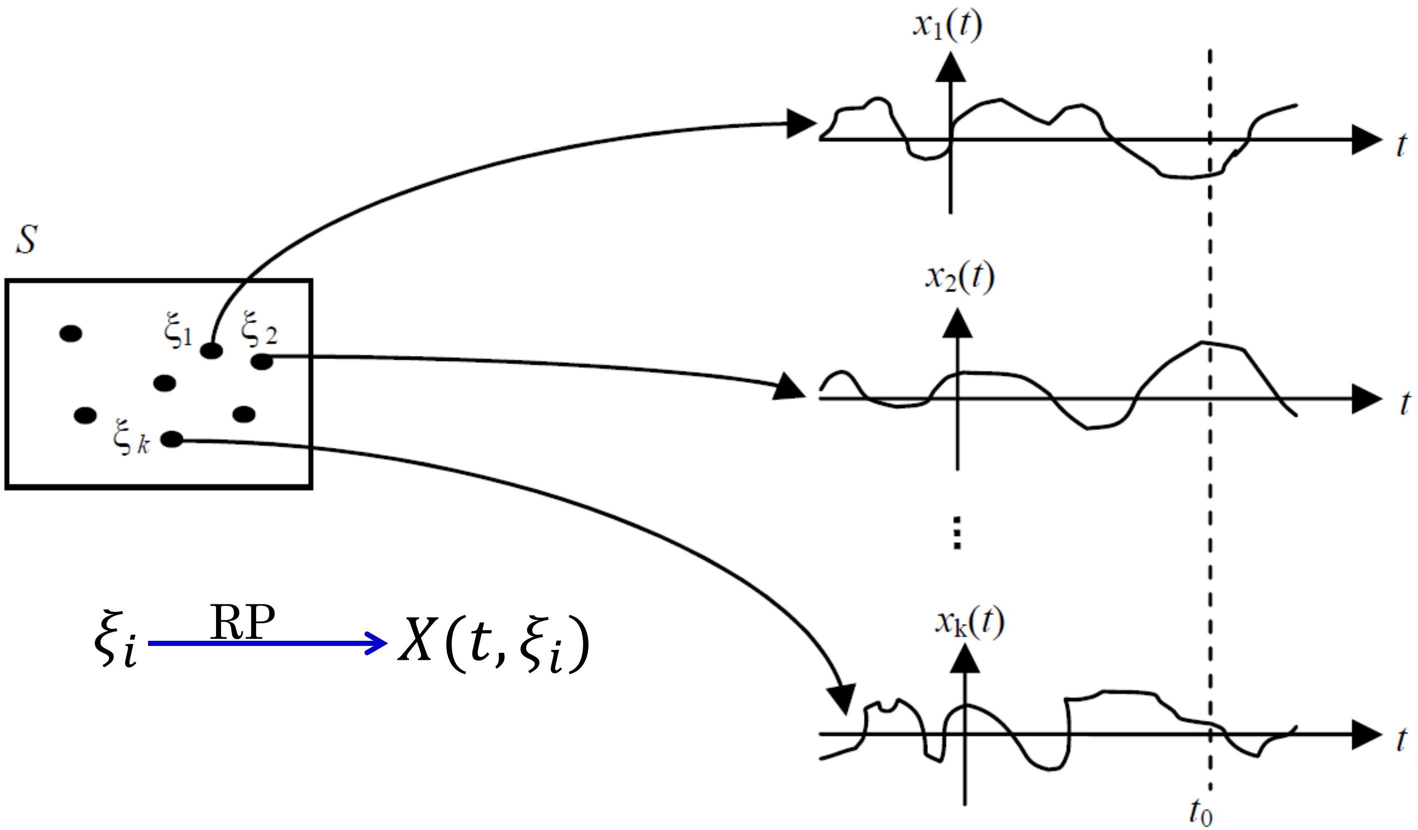
Recall: Random Variable Def



$$\xi_i \xrightarrow{\text{RV}} X(\xi_i)$$

Random process



$$\xi_i \xrightarrow{\text{RP}} X(t, \xi_i)$$

Random process

$$\xi_i \xrightarrow{\text{RP}} X(t, \xi_i)$$

A random process is a function denoted by $X(t, \xi)$

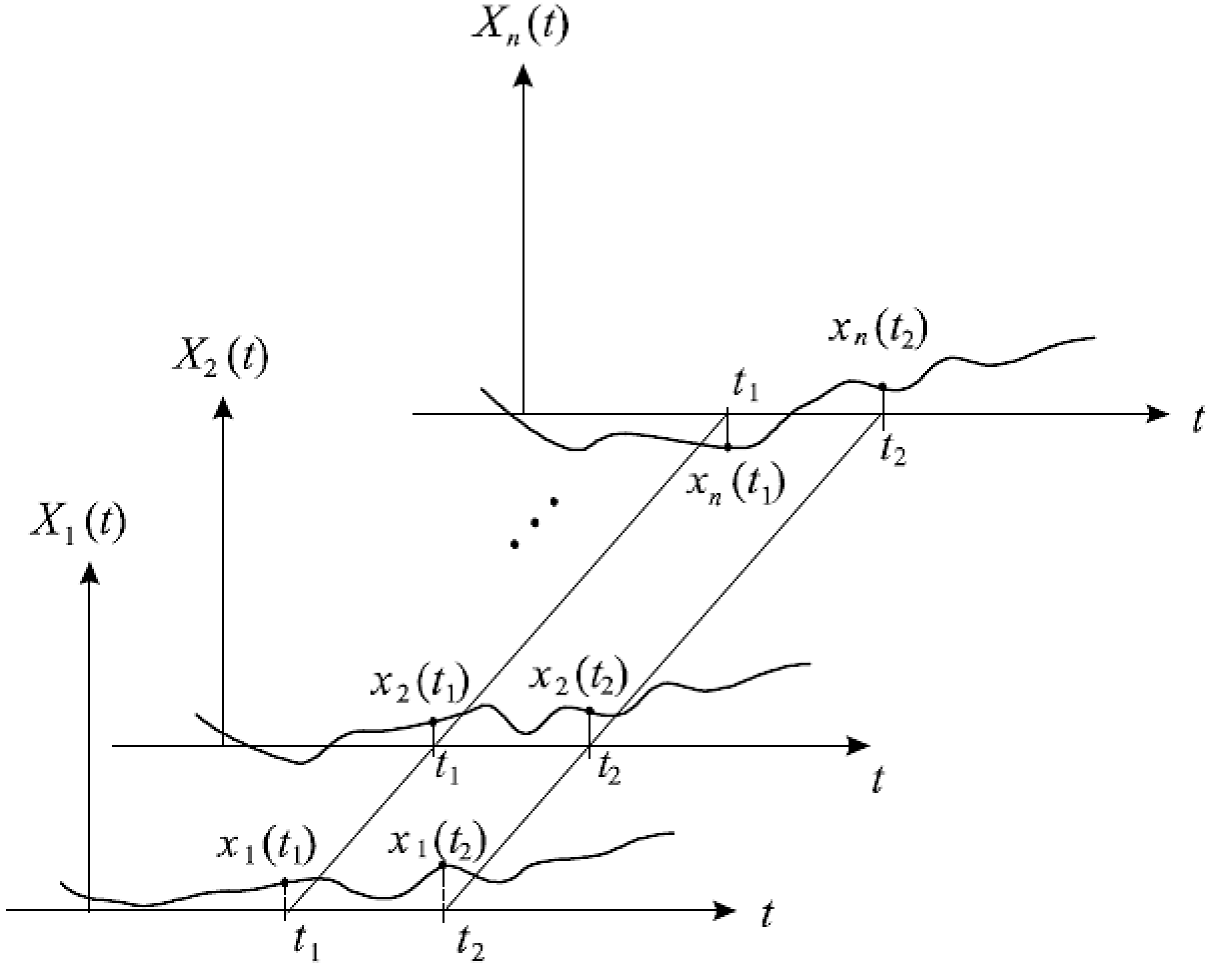
$(a)$for a fixed value of t, $X(t, \xi)$is a random variable,

(b) for a fixed value of ξ , $X(t, \xi)$is a function of time (a realization),

(c) for fixed values of t and ξ , $X(t, \xi)$is a number, and

(d) for varying t and ξ , $X(t, \xi)$ is collection of time histories (ensemble)

Random process

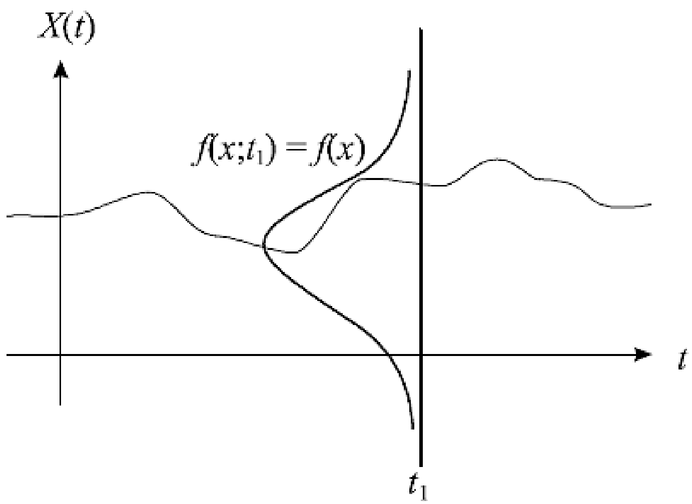


Random process

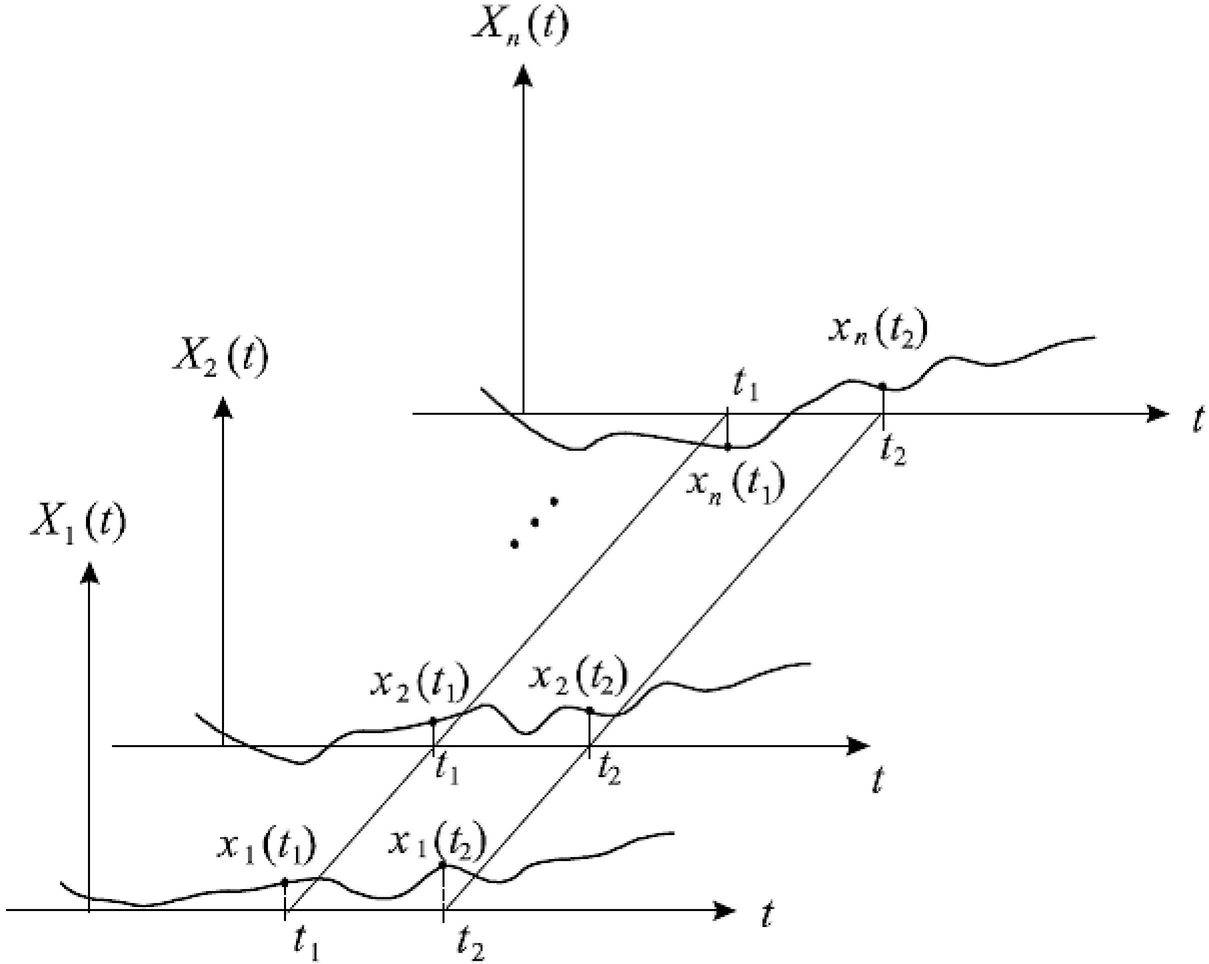
First-order distribution (for a particular value of t)

$$F_X(x;t) = P[X(t_0) \le x]$$

First-order density function $f_X(x;t) = \frac{d}{dx} F_X(x;t)$



2nd Order Averages



2nd Order Averages

2nd order distribution

$$F_X(x_1, x_2; t_1, t_2) = P[X(t_1) \le x_1 \text{ and } X(t_2) \le x_2]$$

2nd order density function

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2; t_1, t_2)$$

Expectations

Ensemble Average

The *mean* of $X(t)$ is defined by

$$\mu_X(t) = E[X(t)]$$

$X(t)$ is treated as a random variable for a fixed value of t.

Autocorrelation

$$R_{xx}(t_1,t_2) = E\left[X(t_1)X(t_2)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_{X_1 X_2}(x_1,x_2;t_1,t_2)\,dx_1 dx_2$$

Autocovariance

$$\begin{aligned} K_X(t,s) = \mathrm{Cov}[X(t),X(s)] &= E\{[X(t)-\mu_X(t)][X(s)-\mu_X(s)]\} \\ &= R_X(t,s)-\mu_X(t)\mu_X(s) \end{aligned}$$

Random process

The random process $X(t)$ is given by

$$X(t) = A\cos(\omega t - \Phi),$$

where A and Φ are random variables with the probability density function,

$$f_{A\Phi}(a,\phi) = \frac{1}{2\pi}\left(1 + (3a-1)\cos\phi\right),$$
$$\text{for} \quad 0 \le \phi \le 2\pi$$
$$\text{and} \quad 0 \le a \le 1.$$

Derive (a) μ_X, (b) σ_X^2, and (c) $R_{XX}(t_1, t_2)$.

Random process

(a) The mean can be found by taking the expected value of $X(t)$ or

$$E\{X(t)\} = E\{A\cos(\omega t - \Phi)\},$$

which can be expanded to

$$E\{X(t)\} = E\{A(\cos\omega t \cos\Phi + \sin\omega t \sin\Phi)\}.$$

Since only A and ϕ are random, $\cos\omega t$ and $\sin\omega t$ can be taken out of the expectation so that

$$E\{X(t)\} = \cos\omega t E\{A\cos\Phi\} + \sin\omega t E\{A\sin\Phi\},$$

Random process

$$E\{X(t)\} = \cos\omega t E\{A\cos\Phi\} + \sin\omega t E\{A\sin\Phi\},$$

where

$$\begin{aligned} E\{A\cos\Phi\} &= \int_0^1\int_0^{2\pi} a\cos\phi f_{A\Phi}(a,\phi)\,d\phi da \\ &= \frac{1}{4} \\ E\{A\sin\Phi\} &= \int_0^1\int_0^{2\pi} a\sin\phi f_{A\Phi}(a,\phi)\,d\phi da \\ &= 0. \end{aligned}$$

Then,

$$E\{X(t)\} = \frac{1}{4}\cos\omega t,$$

which means that $X(t)$ is a nonstationary random process.

Random process

(b) The variance can be found using

$$\sigma_X^2 = E\left\{(X(t) - \mu_X)^2\right\} = E\{X^2\} - \mu_X^2.$$

The root mean square $E\{X^2\}$ is given by

$$\begin{aligned} E\{X^2\} &= E\{A^2(\cos\omega t\cos\Phi + \sin\omega t\sin\Phi)^2\} \\ &= E\{A^2\cos^2\omega t\cos^2\Phi + A^2\sin^2\omega t\sin^2\Phi \\ &\quad +2A^2\cos\omega t\cos\Phi\sin\omega t\sin\Phi\} \\ &= \cos^2\omega t E\{A^2\cos^2\Phi\} + \sin^2\omega t E\{A^2\sin^2\Phi\} \\ &\quad +2\cos\omega t\sin\omega t E\{A^2\cos\Phi\sin\Phi\}. \end{aligned}$$

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Random process

(b) The variance can be found

Each term in the previous equation can be evaluated as follows

$$\begin{aligned} E\left\{A^2\cos^2\Phi\right\} &= \frac{1}{6} \\ E\left\{A^2\sin^2\Phi\right\} &= \frac{1}{6} \\ E\left\{A^2\cos\Phi\sin\Phi\right\} &= 0. \end{aligned}$$

Then,

$$E\left\{X^2\right\} = \frac{1}{6}$$

and the variance equals

$$\begin{aligned} \sigma_X^2 &= R_{XX}(t,t) - \mu_X^2 \\ &= E\left\{X^2\right\} - \mu_X^2 = \frac{1}{6} - \frac{1}{16}\cos\omega^2 t. \end{aligned}$$

Random process

(c) The autocorrelation function $R_{XX}(t_1, t_2)$, by definition, is given by

$$\begin{aligned} R_{XX}(t_1, t_2) &= E\left\{A^2 \cos(\omega t_1 - \Phi)\cos(\omega t_2 - \Phi)\right\} \\ &= \cos\omega t_1 \cos\omega t_2 E\{A^2 \cos^2\Phi\} + \sin\omega(t_1 + t_2) E\left\{A^2 \cos\Phi \sin\Phi\right\} \\ &\quad + \sin\omega t_1 \sin\omega t_2 E\{A^2 \sin^2\Phi\} \\ &= \frac{1}{6}\cos\omega(t_1 - t_2). \end{aligned}$$

Autocorrelation: example

Consider the random process $X(t)$ $\qquad X(t) = Y\cos\omega t \qquad t \geq 0$

where ω is a constant and Y is a uniform r.v. over $(0, 1)$.

(a) Find $E[X(t)]$.

(b) Find the autocorrelation function $R_x(t, s)$ of $X(t)$.

(c) Find the autocovariance function $K_x(t, s)$ of $X(t)$

Autocorrelation: example

(a) $E(Y) = \frac{1}{2}$ and $E(Y^2) = \frac{1}{3}$. Thus,

$$E[X(t)] = E(Y \cos \omega t) = E(Y) \cos \omega t = \frac{1}{2} \cos \omega t$$

(b)
$$R_X(t, s) = E[X(t)X(s)] = E(Y^2 \cos \omega t \cos \omega s)$$
$$= E(Y^2) \cos \omega t \cos \omega s = \frac{1}{3} \cos \omega t \cos \omega s$$

(c)
$$K_X(t, s) = R_X(t, s) - E[X(t)]E[X(s)]$$
$$= \frac{1}{3} \cos \omega t \cos \omega s - \frac{1}{4} \cos \omega t \cos \omega s$$
$$= \frac{1}{12} \cos \omega t \cos \omega s$$

Classification of stochastic process

Strictly stationary

A random process $\{X(t), t \in T\}$ is said to be *stationary* or *strict-sense stationary* if, for all n and for every set of time instants $(t_i \in T, i = 1, 2, \ldots, n\}$,

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n; t_1 + \tau, \ldots, t_n + \tau)$$

Thus both first order and second order distributions are independent of t

$$F_X(x; t) = F_X(x; t + \tau) = F_X(x)$$
$$f_X(x; t) = f_X(x)$$
$$\mu_X(t) = E[X(t)] = \mu$$
$$\mathrm{Var}[X(t)] = \sigma^2$$

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; t_2 - t_1)$$
$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_2 - t_1)$$

Wide sense stationary

If stationary condition of a random process X(t) does not hold for all n but holds for $n \leq k$, then we say that the process X(t)is stationary to order k.

If X(t) is stationary to order 2, then X(t) is said to be wide-sense stationary (WSS) or weak stationary.

1. $E[X(t)] = \mu$ (constant)
2. $R_X(t, s) = E[X(t)X(s)] = R_X(|s - t|)$

Wide sense stationary

- Stationarity of a random process
  is analogous to
  steady state in vibration problems

- One or more of the properties of random process becomes independent of time

- Strong sense stationarity (SSS) : defined with respect to pdf-s

- Wide sense stationarity (WSS) : defined with respect to moments

Stationary SS: Few Theorems

1. If a random process which is stationary to order **n** is also stationary to all orders lower than **n.**

2. If {X(t), $t \in T$} is a strict-sense stationary random process, then it is also WSS.

3. If a random process X(t) is WSS, then it must also be covariance stationary

SSS: Example

Consider a random process $X(t)$ defined by

$$X(t) = U \cos \omega t + V \sin \omega t \qquad -\infty < t < \infty$$

where ω is constant and U and V are r.v.'s.

(a) Show that the condition

$$E(U) = E(V) = 0$$

is necessary for $X(t)$ to be stationary.

(b) Show that $X(t)$ is WSS if and only if U and V are uncorrelated with equal variance; that is,

$$E(UV) = 0 \qquad E(U^2) = E(V^2) = \sigma^2$$

(*a*)

$$\mu_X(t) = E[X(t)] = E(U)\cos\omega t + E(V)\sin\omega t$$

must be independent of t for $X(t)$ to be stationary.

This is possible only if $\mu_x(t) = 0$, that is, $E(U) = E(V) = 0$.

(*b*) If $X(t)$ is WSS, then

$$E[X^2(0)] = E\left[X^2\left(\frac{\pi}{2\omega}\right)\right] = R_{XX}(0) = \sigma_X{}^2$$

But $X(0) = U$ and $X(\pi/2\omega) = V$; thus,

$$E(U^2) = E(V^2) = \sigma_X{}^2 = \sigma^2$$

Using the above result, we obtain

$$\begin{aligned} R_x(t, t+\tau) &= E[X(t)X(t+\tau)] \\ &= E\{(U\cos\omega t + V\sin\omega t)[U\cos\omega(t+\tau) + V\sin\omega(t+\tau)]\} \\ &= \sigma^2\cos\omega\tau + E(UV)\sin(2\omega t + \omega\tau) \end{aligned}$$

Conversely, if $E(UV) = 0$ and $E(U^2) = E(V^2) = \sigma^2$, then from the result of part (*a*) and the above result

$$\mu_x(t) = 0$$

$$R_x(t, t+\tau) = \sigma^2\cos\omega\tau = R_x(\tau)$$

$R_x(\tau)$ (WSS) examples

1) $G(t) = A\cos(\omega_0 t + \phi)$, where ϕ is uniform RV with $\phi \sim U(0, 2\pi)$. Determine the mean and the autocorrelation ?

Ans = $\frac{A^2}{2}\cos(\omega_0\tau)$

2) $G(t) = A\cos(\omega t + \theta)$, where ω and θ are independent RVs with $\theta \sim U(0, 2\pi)$ and $\omega \sim U(\omega_1, \omega_2)$. Determine the mean and the autocorrelation ?

Ans = $\frac{A^2}{2\tau(\omega_2 - \omega_1)}[\sin\omega_2\tau - \sin\omega_1\tau]$

Autocorrelation: Properties

1. It is an even function of τ

$$R_x(\tau) = R_x(-\tau)$$

2. Bounded by its value at origin

$$|R_x(\tau)| \leq R_x(0)$$

3. $R_x(0) = E[X^2]$

4. If X is periodic $R_x(\tau)$ is also periodic

Autocorrelation: Example

A random process $Y(t)$ is given by $Y(t) = X(t) \cos(\omega t + \Phi)$, where $X(t)$, a zero mean wide-sense stationary random process with autocorrelation function $R_X(\tau) = 2e^{-2\lambda|\tau|}$ is modulating the carrier $\cos(\omega t + \Phi)$. The random variable Φ is uniformly distributed in the interval $(0,2\pi)$, and is independent of $X(t)$. We have to find the mean, variance, and autocorrelation of $Y(t)$:

Autocorrelation: Example

Mean. The independence of $X(t)$ and Φ allows us to write

$$E[Y(t)] = E[X(t)]E[\cos(\omega t + \Phi)]$$

and with $E[X(t)] = 0$ and $E[\cos(\omega t + \Phi)] = 0$ $\quad E[Y(t)] = 0$

Variance. Since $X(t)$ and Φ are independent, the variance can be given by

$$\sigma_Y^2 = E[Y^2(t)] = E[X^2(t)\cos^2(\omega t + \Phi)] = \sigma_X^2 E[\cos^2(\omega t + \Phi)]$$

However

$$E[\cos^2(\omega t + \Phi)] = \frac{1}{2}E[1 + \cos(2\omega t + 2\Phi)] = \frac{1}{2} \quad \text{and} \quad \sigma_X^2 = C_X(0) = R_X(0) = 2$$

and hence $\sigma_Y^2 = \sigma_X^2/2 = 1$.

Autocorrelation: Example

Autocorrelation:

$$R_Y(\tau) = E[Y(t)Y(t+\tau)] = E[X(t)\cos(\omega t + \Phi)X(t+\tau)\cos(\omega t + \omega\tau + \Phi)]$$

$$= R_X(\tau)\frac{1}{2}E[\cos(\omega\tau) + \cos(2\omega t + \omega\tau + 2\Phi)]$$

$$= \frac{R_X(\tau)}{2}\cos(\omega\tau) + \frac{R_X(\tau)}{2}E[\cos(2\omega t + \omega\tau + 2\Phi)]$$

$E[\cos(2\omega t + \omega\tau + 2\Phi)] = 0$, and hence

$$R_Y(\tau) = \frac{R_X(\tau)}{2}\cos(\omega\tau) = e^{-2\lambda|\tau|}\cos(\omega\tau)$$

Cross-correlation

1. Two processes X(t) and Y(t) are called jointly stationary

   - if each of them are WSS individually
   - $R_{xy}(t, t+\tau) = R_{xy}(\tau)$
     $R_{yx}(t, t+\tau) = R_{yx}(\tau)$

2. $R_{xy}(\tau)$ and $R_{yx}(\tau)$ are mirror images of each other $R_{xy}(\tau) = R_{yx}(-\tau)$

Applications

Noisy signals

Consider a signal buried in white-noise, i.e. y(t) = s(t) + n(t)

Assume: Noise and signal are uncorrelated and with mean = 0

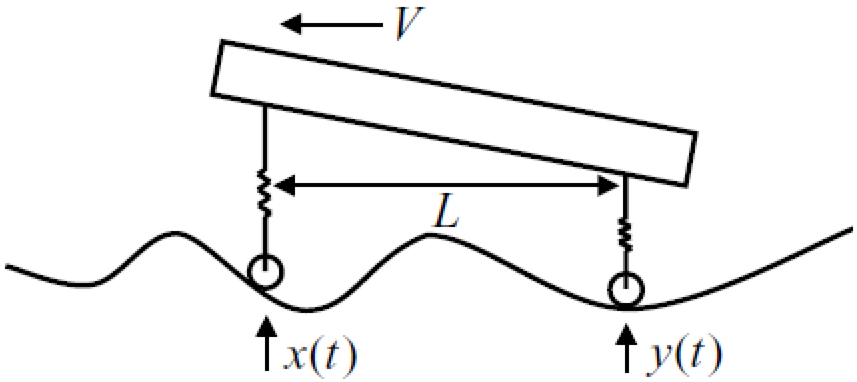
Therefore: $R_{sn}(\tau) = E[s(t)n(t+\tau)] = \mu_s \mu_n$

$$R_{yy}(\tau) = E\left[(s(t)+n(t))(s(t+\tau)+n(t+\tau))\right]$$
$$= E\left[s(t)s(t+\tau)\right] + E\left[n(t)n(t+\tau)\right] + 2\mu_s\mu_n$$

$$R_{yy}(\tau) = R_{ss}(\tau) + R_{nn}(\tau)$$

As Rnn(τ) decays very rapidly, the autocorrelation function of the signal *Rss* (*τ*) will dominate for larger values of *τ*

Application of cross-correlation



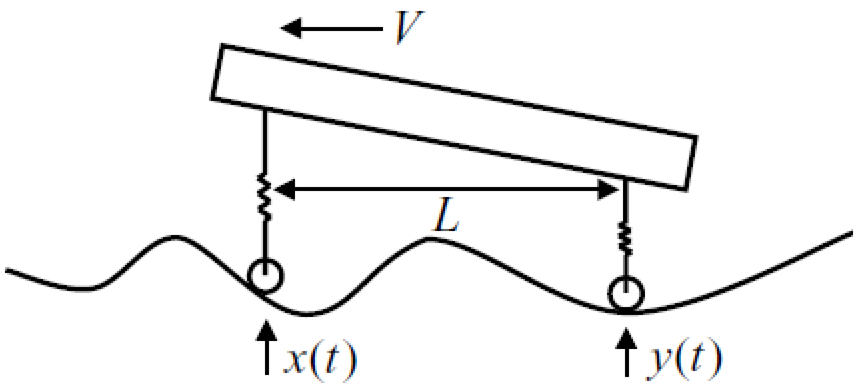
Consider a wheeled vehicle moving over rough terrain as shown in Figure.

- Let the time function (profile) experienced by the leading wheel be $x(t)$ and that by the trailing wheel be $y(t)$
- Let the autocorrelation of x (t) be $R_{xx}(\tau)$
- Assume that the vehicle moves at a constant speed V.

  Then, $y(t) = x(t - \Delta)$ where $\Delta = L/V$

$$R_{xy}(\tau) = E[x(t)y(t+\tau)] = E[x(t)x(t+\tau-\Delta)]$$
$$= R_{xx}(\tau - \Delta)$$

Application of cross-correlation



- Let x (t) and y(t) be observed in presence of white noise ($\sim N(0, \sigma^2)$)

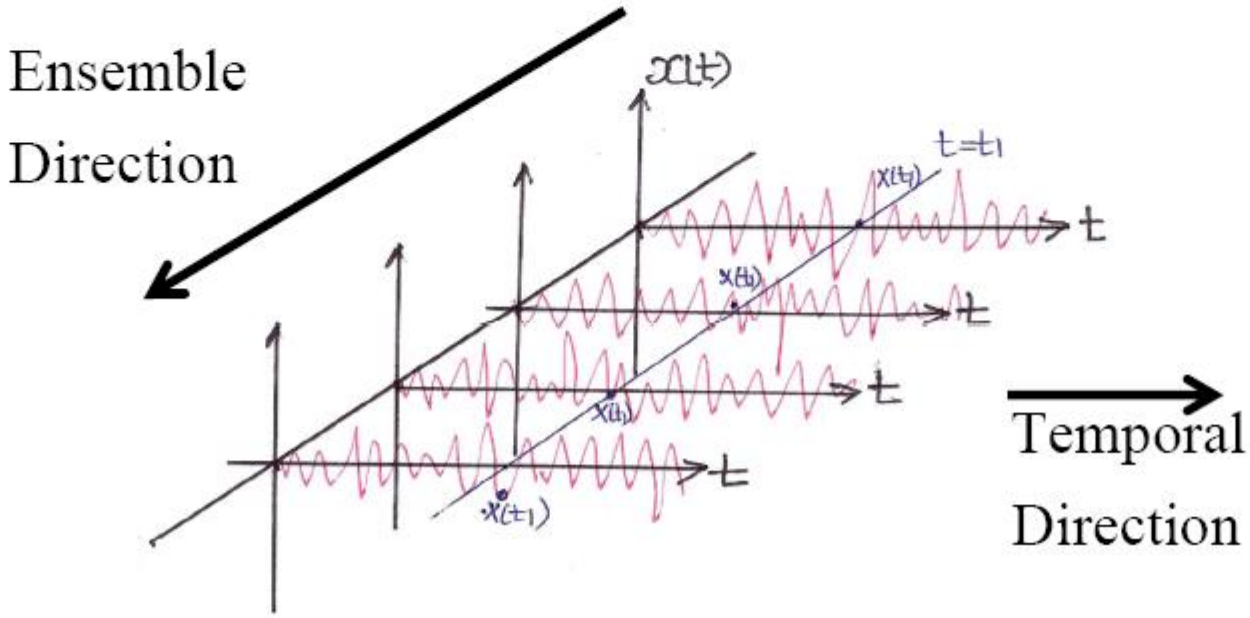
$$x(t) = s(t) + n_x(t)$$
$$y(t) = s(t - \Delta) + n_y(t)$$

The cross-correlation function $R_{xy}(\tau)$ is (assuming zero mean values)

$$R_{xy}(\tau) = E[(s(t) + n_x(t))\left(s(t - \Delta + \tau) + n_y(t + \tau)\right)]$$
$$= E\left[s(t)s(t + \tau - \Delta)\right] = R_{ss}(\tau - \Delta)$$

Erogodicity

Basic idea: Equivalence of temporal and ensemble averages



Erogodicity

A random process is said to be Ergodic if it has the property that the time averages of sample functions of the process are equal to the corresponding statistical or ensemble averages.

$$E[X(t)] = \langle X(t) \rangle = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt$$

The sample autocorrelation can be calculated using the following formula

$$R_X(\tau) = \langle X(t)X(t+\tau) \rangle = \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)\, dt$$

Erogodicity

- Consider a sample of a random process: x (1), x (2),………x (N)
- The sample mean of the sequence could be estimated as:

$$\widehat{m_x}(N) = \frac{1}{N}\sum_{n=0}^{N-1} x_n$$

- Since the sample is a realization of a random process it must have a constant ensemble mean E[X(n)]= m_x

If the sample mean $\widehat{m_x}(N)$ of a WSS converges to m_x in a *mean square sense* as N→ ∞, then the random process is said to be Ergodic in mean

$$\lim_{N\to\infty} \widehat{m_x}(N) = m_x$$

Mean Ergodic Theorem

Mean Ergodic Theorem 1. Let $x(n)$ be a WSS random process with autocovariance sequence $c_x(k)$. A necessary and sufficient condition for $x(n)$ to be ergodic in the mean is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} c_x(k) = 0$$

Mean Ergodic Theorem 2. Let $x(n)$ be a WSS random process with autocovariance sequence $c_x(k)$. Sufficient conditions for $x(n)$ to be ergodic in the mean are that $c_x(0) < \infty$ and

$$\lim_{k\to\infty} c_x(k) = 0$$

Sample autocorrelation of a WSS and Ergodic process

$$r_x(k) = E[\, x(k)x(n-k)]$$

For each k, the autocorrelation is the expected value of the process: $y_k(n) = x(k)x(n-k)$

Using Ergodicity properties, the autocorrelation is finally estimated as :

$$\widehat{r_x}(k,N) = \frac{1}{N}\sum_{n=0}^{N-1} x(k)x(n-k)$$

WSS& Ergodic process: example

Coming back to the random phase sinusoid

$G(t) = A\cos(\omega_0 t + \phi)$, where ϕ is uniform RV with $\phi \sim U(0, 2\pi)$.

$$\langle X(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A\cos(\omega_0 t + \phi)\, dt = 0$$

$$\langle X(t)X(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A^2\cos(\omega_0 t + \omega_0\tau + \phi)cos(\omega_0 t + \phi)dt$$

$$= \frac{A^2}{2}\cos(\omega_0\tau)$$

MATLAB examples

- Autocorrelation of a random phase sinusoid
- Noisy signal
- Time delay problem

FOURIER TRANSFORM

- Extension of Fourier analysis to non-periodic phenomena
- Discrete to continuous
- Skipping essential steps, in the limit $T_p \rightarrow \infty$

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

Fourier transform

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

Inverse Fourier transform

POWER SPECTRUM

Knowing that a random process is composed of energy at many frequencies, we define a random process that is a sum of harmonics, similarly to the Fourier series. Begin first with a random function of one harmonic process, $X(t) = C\cos(\omega t - \phi)$, or equivalently,

$$X(t) = A\cos\omega t + B\sin\omega t,$$

where A and B are independent random variables. Assuming both to be zero mean and identically distributed, we have

$$\begin{aligned} \mu_A = \ & \mu_B = \ 0 \\ \sigma_A^2 = \ & \sigma_B^2 = \ \sigma^2. \end{aligned}$$

POWER SPECTRUM

The autocorrelation is given by

$$\begin{aligned} R_{XX}(\tau) &= E\{X(t)\,X(t+\tau)\} \\ &= E\{(A\cos\omega t + B\sin\omega t)(A\cos\omega[t+\tau] + B\sin\omega[t+\tau])\}. \end{aligned}$$

Expanding the product and utilizing trigonometric identities results in

$$R_{XX}(\tau) = \sigma^2 \cos\omega\tau.$$

Suppose that the frequency content of the random process is expanded,

$$\begin{aligned} X(t) &= \sum_{k=1}^{m} X_k(t) \\ &= \sum_{k=1}^{m} (A_k \cos\omega_k t + B_k \sin\omega_k t), \end{aligned}$$

where we make the same assumptions as were made above about A and B.

POWER SPECTRUM

Suppose that the frequency content of the random process is expanded,

$$\begin{aligned} X(t) &= \sum_{k=1}^{m} X_k(t) \\ &= \sum_{k=1}^{m} \left(A_k \cos \omega_k t + B_k \sin \omega_k t \right), \end{aligned}$$

where we make the same assumptions as were made above about A and B. Following the same procedure as for the above single-frequency process, we find

$$R_{XX}(\tau) = \sum_{k=1}^{m} R_{X_k X_k}(\tau) = \sum_{k=1}^{m} \sigma_k^2 \cos \omega_k \tau.$$

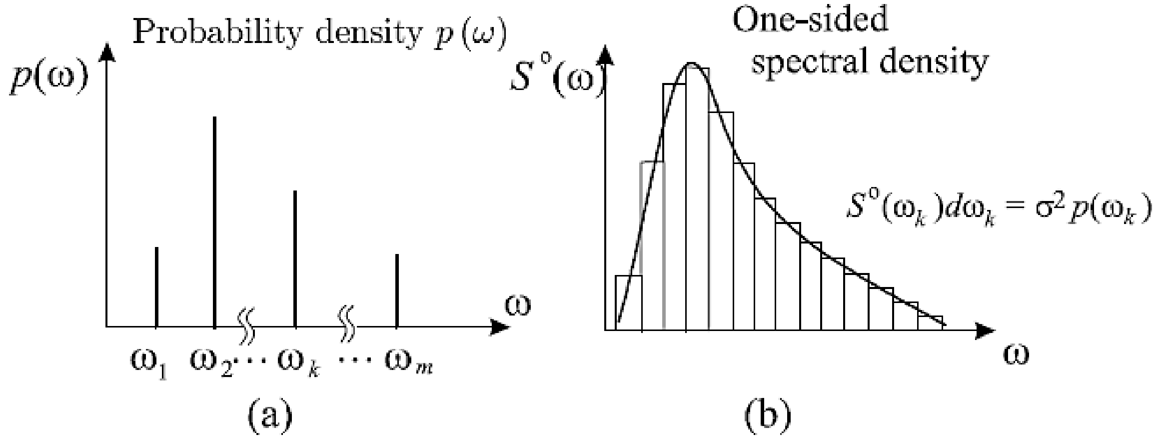
The total variance for the process is found by recalling that

$$\sigma^2 = E\left\{X^2(t)\right\} - \mu_X^2 = R_{XX}(0),$$

the last equality being true for the case where the mean equals zero.

$$\sigma^2 = \sum_{k=1}^{m} \sigma_k^2.$$

POWER SPECTRUM

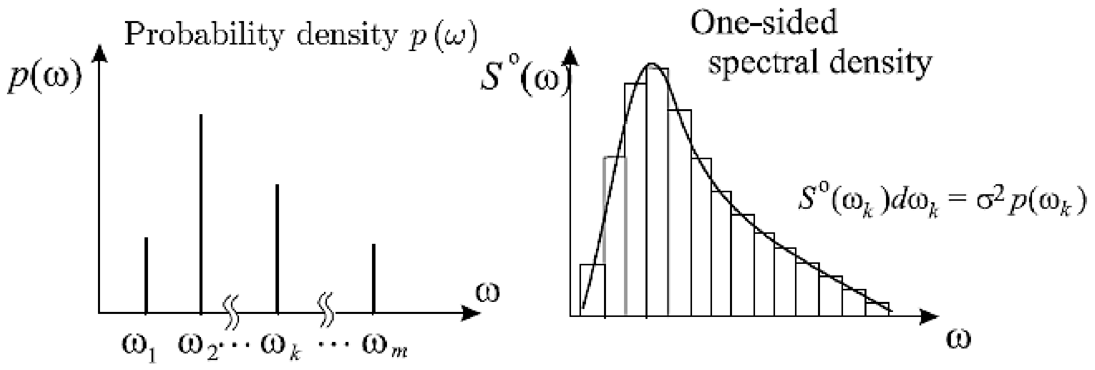


Each frequency component ω_k contributes σ_k^2 to the total variance σ^2. The fraction of the total is given by the ratio σ_k^2/σ^2, which can be defined as $p(\omega_k)$ as shown in Figure Note that $\sum_{k=1}^{m} p(\omega_k) = 1$. Then,

$$R_{XX}(\tau) = \sigma^2 \sum_{k=1}^{m} p(\omega_k) \cos \omega_k \tau,$$

where $p(\omega_k)$ acts as a weighting function. The above implies that $p(\omega_k)$ behaves like a probability density.

POWER SPECTRUM



(a) (b)

Suppose the frequency spectrum becomes very broad, including many frequencies, that is $m \to \infty$, resulting in a continuous frequency spectrum. Define $d\omega = \omega_{k+1} - \omega_k$. In an analogous manner to how we proceeded from a discrete to a continuous probability density function, replace $\sigma^2 p(\omega_k)$ by $S^o(\omega)\, d\omega$, and the sum above by an integral over the frequency range,

$$R_{XX}(\tau) = \int_0^\infty S^o(\omega) \cos \omega\tau \, d\omega.$$

$S^o(\omega)$ is called the *one-sided spectral density* of the random process because it distributes the variance of the random process as a density across the frequency spectrum. The one-sided spectral density is shown in Figure

WIENER KHINCHINE THEOREM

$$S_{XX}(\omega) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau} d\tau$$

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega)\, e^{i\omega\tau} d\omega.$$

PROPERTIES

Since $R_{XX}(\tau) = R_{XX}(-\tau)$, $S_{XX}(\omega)$ is not a complex function but a real even function,

$$S_{XX}(\omega) = S_{XX}(-\omega)\,.$$

For $\tau = 0$,

$$\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = R_{XX}(0) = E\{X^2(t)\} \geq 0.$$

On physical grounds, then, it can be argued that since the area under the power spectral density equals σ_X^2 for a zero-mean random process, it must be a positive quantity for any $\Delta\omega$, that is, $S_{XX}(\omega) \geq 0$.

PROPERTIES

For $\tau = 0$,

$$\int_{-\infty}^{\infty} S_{XX}(\omega)\, d\omega = R_{XX}(0) = E\{X^2(t)\} \geq 0.$$

The above integral represents average or, mean-square power of the process X(t)

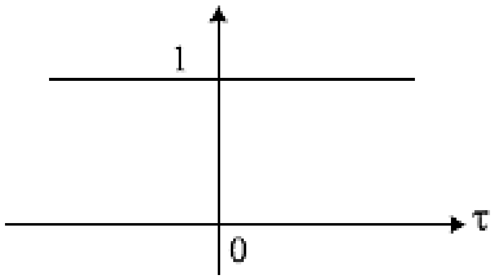
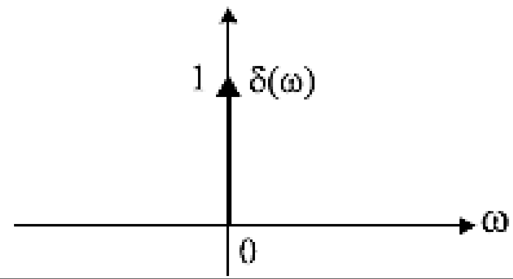
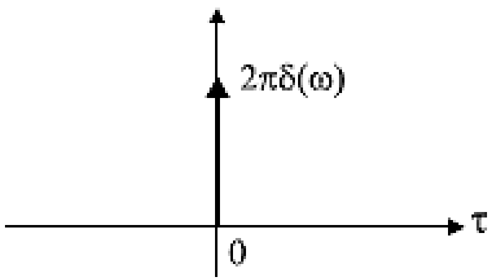
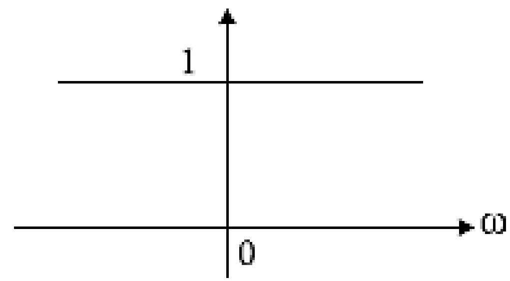
For an ergodic process, the expected value can be written as

$$E\{X^2(t)\} \simeq \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} X^2(t)dt,$$

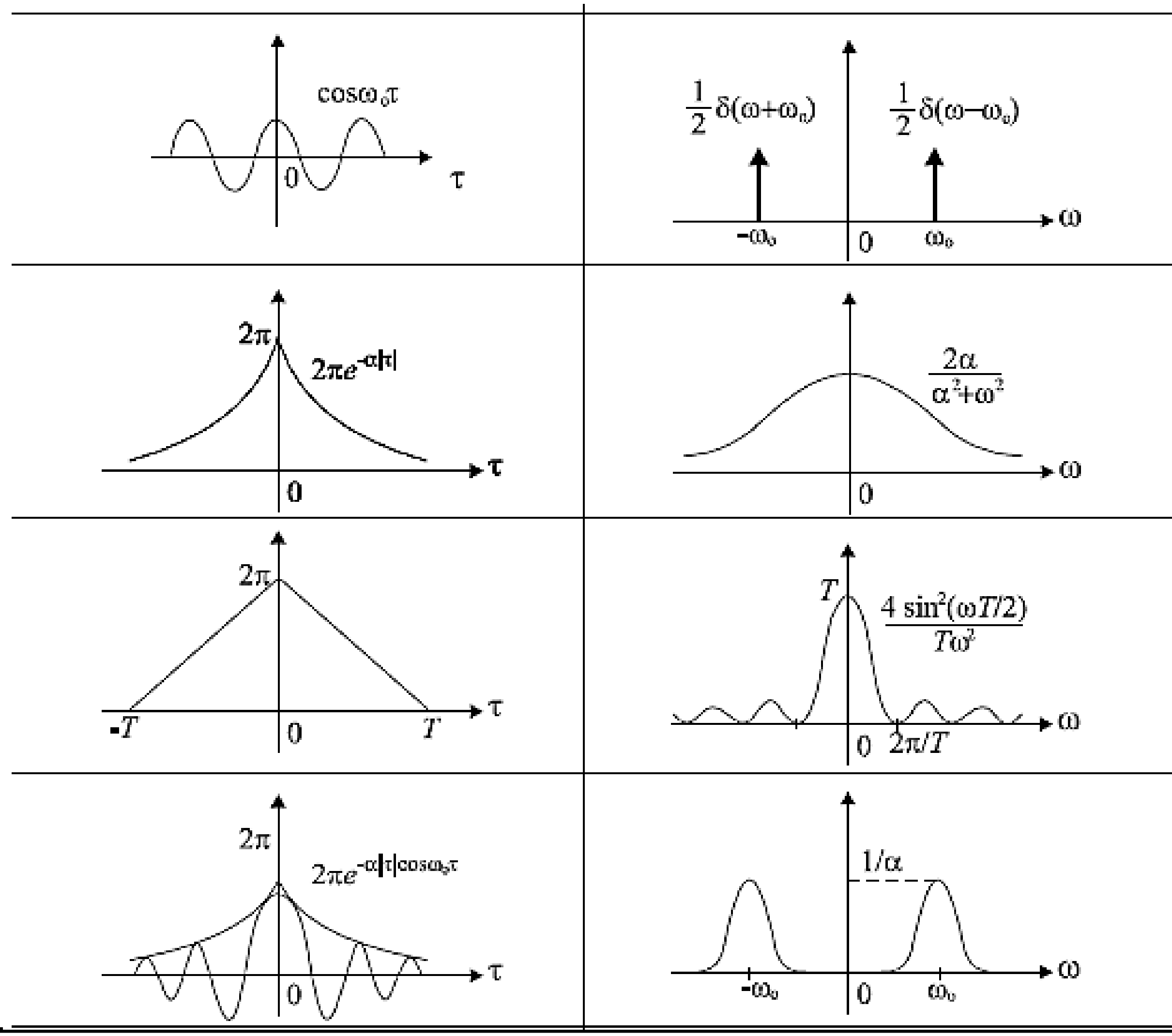
Which is the total energy over the total time or the average power.

Therefore, power spectrum is a measure of the energy

Certain important $R_x(\tau) \leftrightarrow S_x(\omega)$

$R(\tau) = \int_{-\infty}^{\infty} S(\omega) e^{i\omega\tau} d\omega$	$S(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R(\tau) e^{-i\omega\tau} d\tau$
	
	

Certain important $R_x(\tau) \leftrightarrow S_x(\omega)$



Example

Consider for example $R_x(\tau) = \sin \omega_0 \tau$

The spectral density is related to the autocorrelation function by Equation

$$S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} R_{XX}(\tau)\, e^{-i\omega\tau} d\tau.$$

If the autocorrelation is a pure sine function, $S_{XX}(\omega)$ is given by the integral,

$$S_{XX}(\omega) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \sin \omega_o \tau\, e^{-i\omega\tau} d\tau.$$

Example

Consider for example $R_x(\tau) = \sin \omega_0 \tau$

Using the Euler identity, $\sin \omega_o \tau = [\exp(i\omega_o \tau) - \exp(-i\omega_o \tau)]/2i$, the spectral density becomes

$$
\begin{aligned}
S_{XX}(\omega) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2i} [\exp(i\omega_o \tau) - \exp(-i\omega_o \tau)] \, e^{-i\omega\tau} d\tau \\
&= \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{2i} \left(e^{-i(\omega-\omega_o)\tau} - e^{-i(\omega+\omega_o)\tau} \right) d\tau \\
&= \frac{1}{2i} [\delta(\omega - \omega_o) - \delta(\omega + \omega_o)] .
\end{aligned}
$$

This example shows that a sine function cannot be a valid autocorrelation function. Why ?

Units of Power spectral density

Units of PSD: $\dfrac{[\text{Units of } X(t)]^2}{\text{frequency}}$

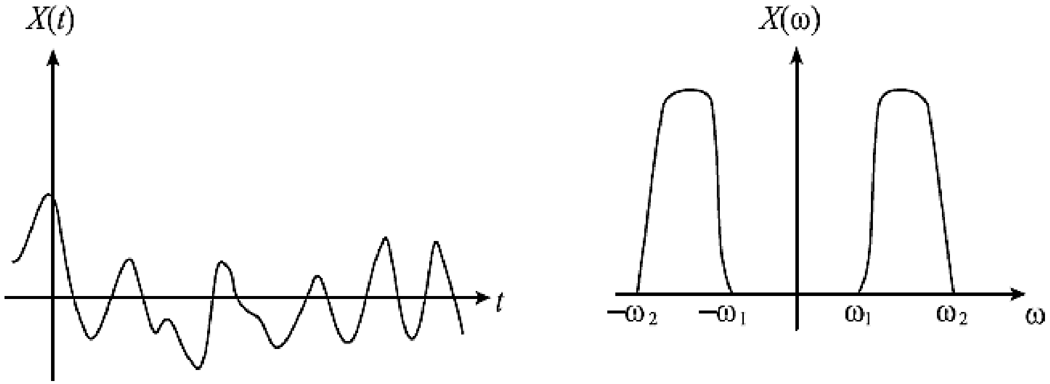
Ex: $X(t)$ is displacement

Units of PSD : $\dfrac{\text{m}^2}{\text{Hz}}$ or $\dfrac{\text{m}^2}{(\text{rad/s})}$

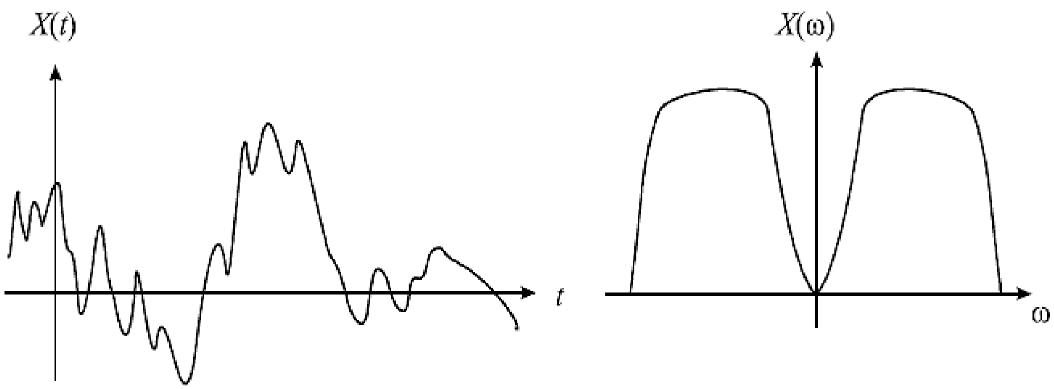
Similarly, if X(t) is acceleration

Units of PSD: $\dfrac{(\text{m/s}^2)^2}{\text{Hz}}$ or $\dfrac{(\text{m/s}^2)^2}{(\text{rad/s})}$

Narrow band & broad band processes



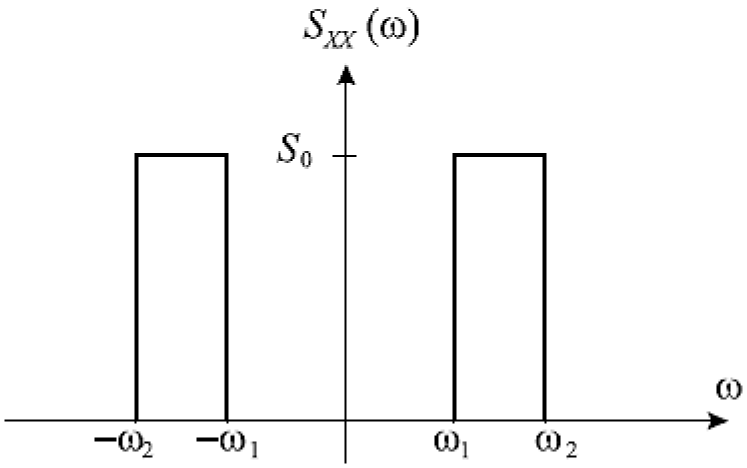
Narrow-band random process $X(t)$ in time and frequency $X(\omega)$ domains.



Broad-band random process $X(t)$ in time and frequency $X(\omega)$ domains

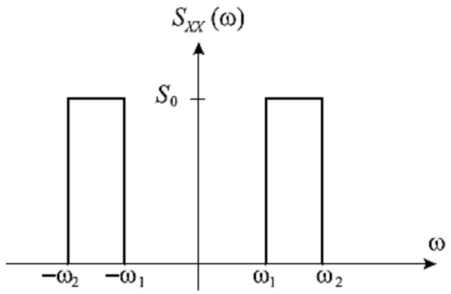
Narrow band & broad band processes

A narrow band spectrum can be expressed as a flat spectrum S_0 in the frequency band $[\omega_1\ \omega_2]$



Narrow band & broad band processes

The autocorrelation function for such a process is evaluated as follows,



$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(\omega) e^{i\omega\tau} d\omega$$
$$= 2\int_{\omega_1}^{\omega_2} S_0 \cos\omega\tau \, d\omega,$$

where the real part of the complex exponential is retained having made use of the symmetry of the power spectrum function. The integral is evaluated to give

$$R_{XX}(\tau) = 2\frac{S_0}{\tau}\left(\sin\omega_2\tau - \sin\omega_1\tau\right). \quad (1)$$

Note that the autocorrelation function consists of two harmonic functions at frequencies ω_1 and ω_2. When the frequencies are close to each other, beating is observed. This is clearer when Equation (1) is written as

$$R_{XX}(\tau) = 4\frac{S_0}{\tau}\cos\left\{\left(\frac{\omega_1+\omega_2}{2}\right)\tau\right\}\sin\left\{\left(\frac{\omega_2-\omega_1}{2}\right)\tau\right\}.$$

Narrow band & broad band processes

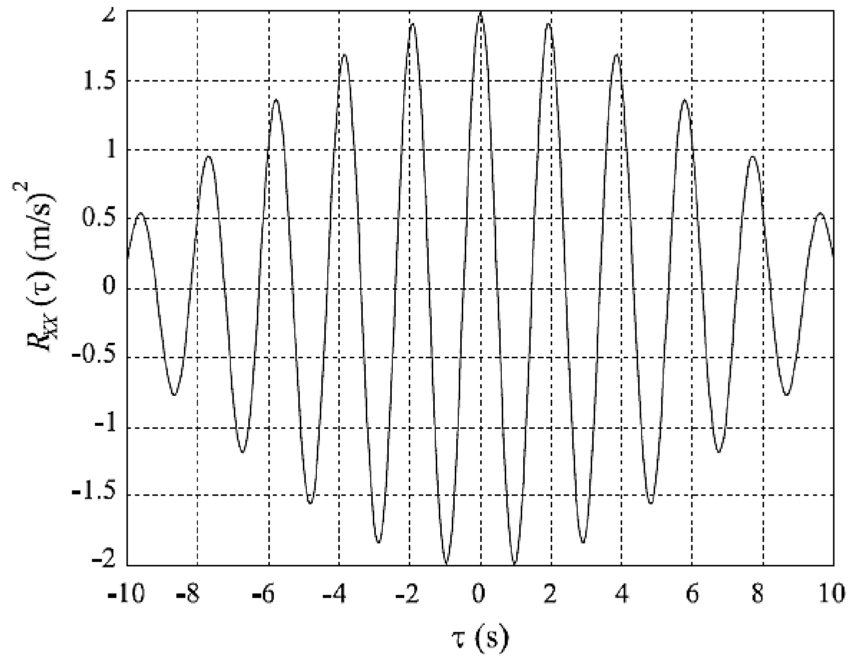


Figure : The autocorrelation function for an ideal narrow-band process. $S_o = 2$ m^2/s, $\omega_1 = 3$ rad/s, and $\omega_2 = 3.5$ rad/s.

Broad band processes

A broad band process is one that contains significant energy for a wider range of frequencies

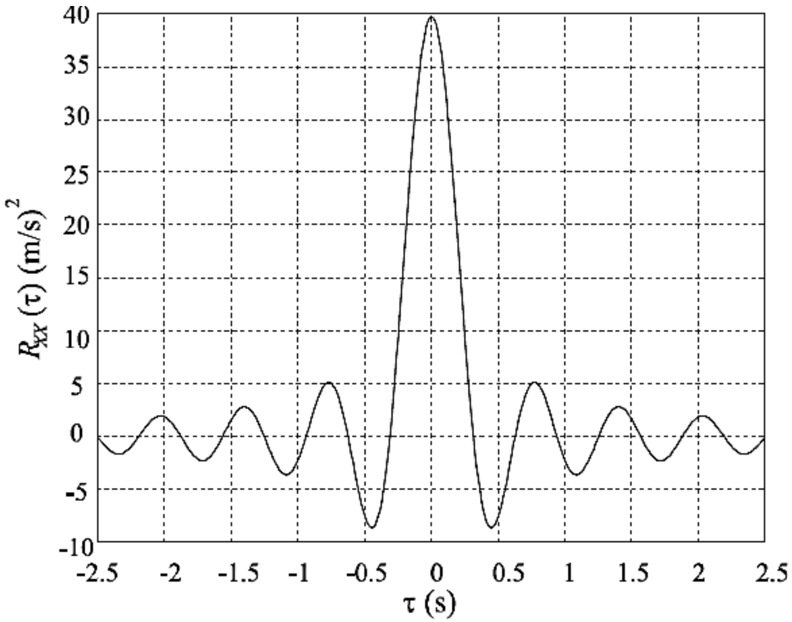


Figure : The autocorrelation function for an ideal narrow-band process $S_0 = 2\ \mathrm{m}^2/\mathrm{s}$, $\omega_1 = 0$ rad/s, and $\omega_2 = 10$ rad/s.

White noise process

What happens when $\boldsymbol{\omega_1} = 0$ and Lim $\boldsymbol{\omega_2} \to \infty$

$$\lim_{\omega_1 \to 0} 4\frac{S_0}{\tau}\cos\left\{\left(\frac{\omega_1+\omega_2}{2}\right)\tau\right\}\sin\left\{\left(\frac{\omega_2-\omega_1}{2}\right)\tau\right\} = 2S_0\frac{\sin\omega_2\tau}{\tau}$$

$$\begin{aligned} R_{XX}(\tau) &= \lim_{\omega_2\to\infty} 2S_0\frac{\sin\omega_2\tau}{\tau} \\ &= 2\pi S_0\delta(\tau) \end{aligned}$$

Equation can be confirmed using the definition of the spectral density given in Equation

$$S_{XX}(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 2\pi S_0\delta(\tau)e^{-i\omega\tau}d\tau = S_0$$

White noise process

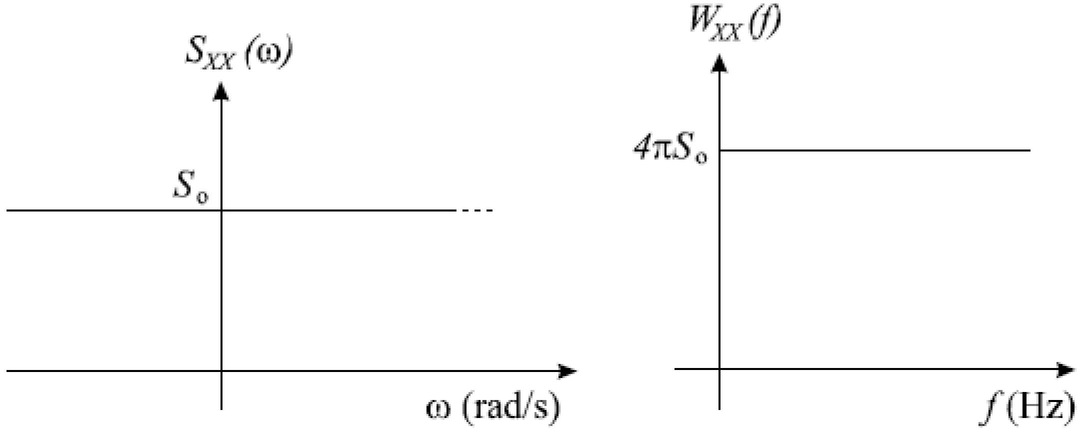


Figure : Two-sided and one-sided white noise spectra.