CE 513: Statistical Methods in Civil Engineering

Lectures-2 & 3: Probability

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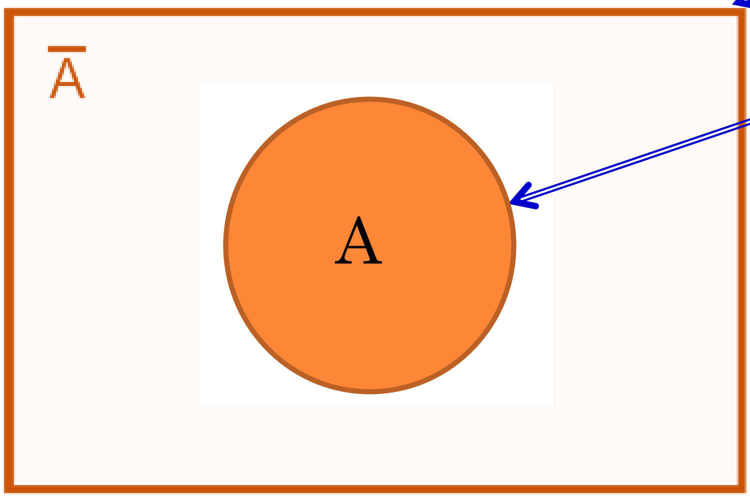
SET THEORY

EVENTS



Every subset of S is an event, including S and the null set Ø

Then

S = the certain event

Ø = the impossible event

SET THEORY

Equality:

Two sets A and B are equal, denoted A =B, if and only if

$$A \subset B \text{ and } B \subset A.$$

Complementation:

Suppose $A \subset S$. The complement of set A, denoted by $\bar{A}$, is the set containing all elements in S but not in A

Union:

The union of sets A and B, denoted **A U B**, is the set containing all elements in either A or B or both

SET THEORY

Intersection:

The intersection of sets A and B, denoted by $\boldsymbol{A \cap B}$, is the set containing all elements in both A and B.

Difference:

The difference of sets A and B, denoted A \ B, is the set containing all elements in A but not in B .

$$A \setminus B = A \cap \bar{B}$$

SET THEORY

Null Set:

The set containing no element is called the null set , denoted by Ø.

Disjoint Sets:

Two sets A and B are called disjoint or mutually exclusive if they contain no common element

SET THEORY

The definitions of the union and intersection of two sets can be extended to any finite number of sets as follows:

$$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \cdots \cup A_n$$

$$= \{\zeta : \zeta \in A_1 \text{ or } \zeta \in A_2 \text{ or } \cdots \zeta \in A_n\}$$

$$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \cdots \cap A_n$$

$$= \{\zeta : \zeta \in A_1 \text{ and } \zeta \in A_2 \text{ and } \cdots \zeta \in A_n\}$$

SET THEORY

If A and B are events in S, then

$$\bar{A} = \text{the event that } A \text{ did not occur}$$

$$A \cup B = \text{the event that either } A \text{ or } B \text{ or both occurred}$$

$$A \cap B = \text{the event that both } A \text{ and } B \text{ occurred}$$

Similarly, if $A_1, A_2, \ldots, A_n$ are a sequence of events in S, then

$$\bigcup_{i=1}^{n} A_i = \text{the event that at least one of the } A_i \text{ occurred}$$

$$\bigcap_{i=1}^{n} A_i = \text{the event that all of the } A_i \text{ occurred}$$

SET THEORY

Size of Set:

When sets are countable, the size (or cardinality) of set A, denoted |A|, is the number of elements contained in A. When sets have a finite number of elements, it is easy to see that size has the following properties:

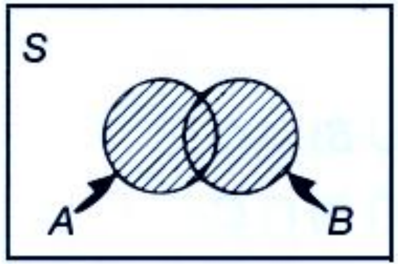
(*i*) If $A \cap B = \varnothing$, then $|A \cup B| = |A| + |B|$

(*ii*) $|\varnothing| = 0$.

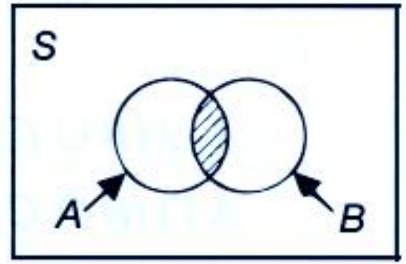
(*iii*) If $A \subset B$, then $|A| \leq |B|$.

(*iv*) $|A \cup B| + |A \cap B| = |A| + |B|$

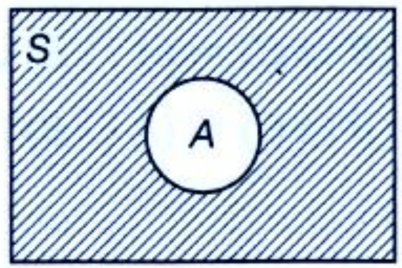
Venn-diagrams



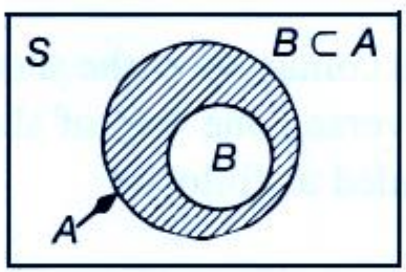
(*a*) Shaded region: $A \cup B$



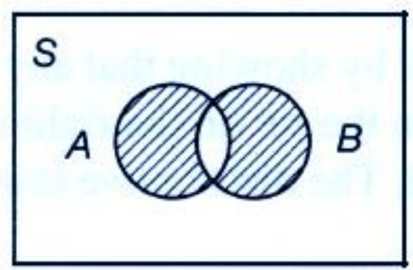
(*b*) Shaded region: $A \cap B$



(c) Shaded region: $\overline{A}$



(*a*) $A \cap \overline{B} = A \setminus B$



(*b*) Shaded area: $A \,\Delta\, B$

SET THEORY: Useful identities

$$\overline{S} = \varnothing$$
$$\overline{\varnothing} = S$$

$$S \cup A = S$$
$$S \cap A = A$$

$$A \backslash B = A \cap \overline{B}$$
$$S \backslash A = \overline{A}$$
$$A \backslash \varnothing = A$$
$$A \,\Delta\, B = (A \cap \overline{B}) \cup (\overline{A} \cap B)$$

$$A \cup \overline{A} = S$$
$$A \cap \overline{A} = \varnothing$$

$$A \cup \varnothing = A$$
$$A \cap \varnothing = \varnothing$$

SET THEORY

Commutative Laws

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

Associative Laws

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

Distributive Laws

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

De Morgan's Laws

$$\overline{A \cup B} = \bar{A} \cap \bar{B}$$

$$\overline{A \cap B} = \bar{A} \cup \bar{B}$$

FIELDS

Fields:

A family $\mathcal{F}$ of subsets of a non-empty set Ω is called a *field* on Ω if

(a) $\Omega \in \mathcal{F}$;

(b) if $A \in \mathcal{F}$, then $\Omega \setminus A \in \mathcal{F}$;

(c) if $A, B \in \mathcal{F}$, then $A \cup B \in \mathcal{F}$.

The elements of a field are called events

- The first condition implies that the set of all outcomes is an event
- The 2^{nd} condition means that the complement of an event is an event
- The 3^{rd} condition implies that the union of 2 events is an event

Properties of fields

$\emptyset \in \mathcal{F}$;

If $A, B, C \in \mathcal{F}$, then $A \cup B \cup C \in \mathcal{F}$;

For each natural number n, if $A_1, \ldots, A_n \in \mathcal{F}$, then $A_1 \cup \cdots \cup A_n \in \mathcal{F}$;

If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$;

For each natural number n, if $A_1, \ldots, A_n \in \mathcal{F}$, then $A_1 \cap \cdots \cap A_n \in \mathcal{F}$;

If $A, B \in \mathcal{F}$, then $A \setminus B \in \mathcal{F}$;

If $A, B \in \mathcal{F}$, then their symmetric difference $A \triangle B$ also belongs to $\mathcal{F}$.

SIGMA FIELDS

The assumption of equally likely events as we will study later leads to a lot of problems in computing probability particularly when of possible outcomes of a random experiment is infinite.

That is precisely the reason why modeling the event space as ordinary fields is not adequate and the notion of sigma field becomes particularly important.

SIGMA FIELDS

A family $\mathcal{F}$ of subsets of a non-empty set Ω is called a *σ-field* (sigma-field) on Ω if

(a) $\Omega \in \mathcal{F}$;

(b) if $A \in \mathcal{F}$, then $\Omega \setminus A \in \mathcal{F}$;

(c) if $A_1, A_2, \ldots \in \mathcal{F}$, then $\bigcup_{i=1}^{\infty} A_i \in \mathcal{F}$.

- ✓ First condition implies that the set of all outcomes is an event
- ✓ 2nd condition means that the complement of an event is an event
- ✓ 3rd condition implies that the union of a sequence of events is an event

SIGMA FIELDS

Properties:

A σ-field is also a field

If a field $\mathcal{F}$ is finite, then it is also a σ- field

A family of all subsets of Ω is a σ- field

If A and B are in sigma field $\mathcal{F}$, then $A \cap B, A \backslash B, A \Delta B$ are also in $\mathcal{F}$

More on σ- fields/algebras

σ-algebras are a subset of algebras in the sense that all σ-algebras are algebras, but not vice versa.

Algebras only require that they are closed under pairwise unions while σ-algebras must be closed under *countably* infinite unions.

Roll of a die: The sample space is $\Omega = \{1, 2, 3, 4, 5, 6\}$

In probability space, the σ-algebra is σ (Ω), also called the σ-algebra generated by Ω.

Take the elements of Ω and generate the "extended set" consisting of all unions, compliments, compliments of unions, unions of compliments, etc. include Φ

With this "extended set“ and the result is σ (Ω), is denoted by Σ.

Example

Let $S = \{1,2,3,4,5,6\}$

Then the collection of the following subsets of S, $\mathscr{F} = \{S, \varnothing, (1,2,3), (4,5,6)\}$ is a field since it satisfies $(1,2,3) \cup (4,5,6) = S$, $(\overline{1,2,3}) = (4,5,6)$

However, the collection of the following subsets of S,

$\mathcal{F}_1 = \{S, \Phi, (1,2,3), (4,5,6), (2)\}$

is not a field because $(2) \cup (4,5,6) = (2,4,5,6)$ is not in the field

But we can adjoin the missing sets and make $\mathcal{F}_1$ into a field. This is known as <u>*completion*</u>. In the example above, if we adjoin the collection of sets

$\boldsymbol{\mathcal{F}_2 = \{(2,4,5,6), (1,3)\}}$ to $\mathcal{F}_1$, then

$\mathcal{F}_1 \cup \mathcal{F}_2 = \{S, \Phi, (1,2,3), (4,5,6), (2), (2,4,5,6), (1,3)\}$ is a field

Borel σ- fields

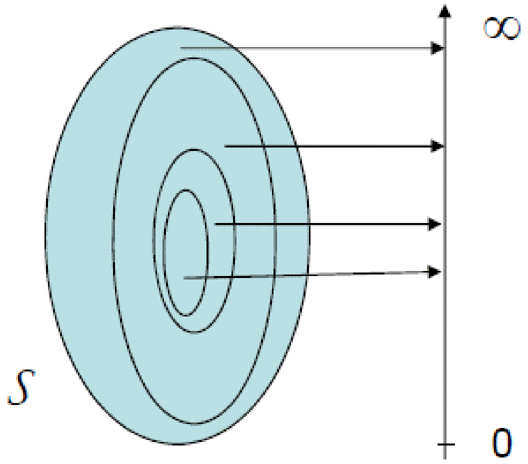
- In many random experiments the outcome is a real number
- We may be interested in finding probability that it belongs to a given interval (a, b)
- To consider all such events, we need a σ-field of the subsets of the real number line $\mathbb{R}$, containing all intervals. This makes the

σ-field very large especially when we consider all possible intervals.

- Many σ-fields may contain the subsets of $\{A_i\}$, *but the smallest σ-field containing the subsets of $\{A_i\}$, is called the Borel σ-field.* The smallest σ-field for S by itself is $F = \{S, \Phi\}$

A brief intro to measure

Measure on a set S: A systematic way to assign a positive number to each suitable subset of S, intuitively interpreted as its *size*. In a sense, it generalizes the concepts of length, area, volume



Examples of measures:

Counting measure: $\mu(S)$ = number of elements in S

Lebesgue measure : $\mu(S)$ = conventional length of S

That is, if S = $[a, b]$ $\Rightarrow \mu(S) = \lambda[a, b] = b - a$.

A brief introduction to measure

A pair (X, Σ) is a ***measurable space*** if X is a set and Σ is a nonempty σ-algebra of the subsets of X.

A measurable space allows us to define a function that assigns real-numbered values to the abstract elements of Σ

Definition:

Let (X, Σ) be a measurable space.

A set function **μ** defined on Σ is called a *measure iff :*

1. $0 \leq \mu(A) \leq \infty$ for any $A \in \Sigma$.
2. $\mu(\Phi) = 0$.
3. (σ-additivity). For any sequence of pairwise disjoint sets $\{A_n\} \in \Sigma$ S.T. $\bigcup_{n=1}^{\infty} A_n \in \Sigma$ we have $\mu(\bigcup_{n=1}^{\infty} A_n) = \sum_{n=1}^{\infty} \mu(A_n)$

Probability measure

A triplet (X, Σ, μ) is a *measure* space if (X, Σ) is a measurable space and $\mu: \Sigma \rightarrow [0; \infty)$ is a measure

• If $\mu(X) = 1$, then μ is a *probability measure*, which we usually use notation P, and the measure space is a *probability space*.

- A measure space (Ω, Σ, μ) is called finite if $\mu(\Omega)$ is a finite real number
- A measure μ is called *σ-finite* if Ω can be decomposed into a countable union of measurable sets of finite measure.

PROBABILITY SPACE

An assignment of real numbers to the events defined in an event space F leads to probability measure P.

Consider a random experiment with a sample space S, and let A be a particular event defined in F.

The probability of the event A is denoted by P(A). Thus, the probability measure is a function defined over F. The triplet (S, F, P) is known as the probability space.

FINITELY ADDITIVE PROBABILITY

Let $\mathcal{F}$ be a field of subsets of Ω. We call a function $P : \mathcal{F} \to [0,1]$ a *finitely additive probability measure* if

(a) $P(\Omega) = 1$;

(b) (*additivity*) for any $A, B \in \mathcal{F}$ such that $A \cap B = \emptyset$

$$P(A \cup B) = P(A) + P(B).$$

We say that $P(A)$ is the *probability* of an event A.

Example

Is P a finitely additive probability measure?

$\Omega = \{1, 2\}$, $\mathcal{F} = \{\emptyset, \Omega, \{1\}, \{2\}\}$,
$P(\emptyset) = 0$, $P(\Omega) = 1$, $P(\{1\}) = \frac{1}{3}$, $P(\{2\}) = \frac{2}{3}$.

We check condition (b) of Definition . The empty set does not contribute to either side of the equality in (b):

$$P(A \cup \emptyset) = P(A) = P(A) + 0 = P(A) + P(\emptyset).$$

The full set Ω is not disjoint with any set except the empty one, and this is covered by the above argument. We only need to consider the case when $A = \{1\}$ and $B = \{2\}$. Then, $P(A \cup B) = P(\Omega) = 1$, which is equal to $P(A) + P(B)$. So P is a finitely additive probability measure.

Example

$\Omega = \{1, 2, 3, 4\}$, $\mathcal{F} = \{\emptyset, \Omega, \{1, 2\}, \{3, 4\}\}$,
$P(\emptyset) = 0$, $P(\Omega) = 1$, $P(\{1, 2\}) = \frac{5}{8}$, $P(\{3, 4\}) = \frac{5}{8}$.

Is P a finitely additive probability measure ?

Let $A = \{1, 2\}$ and $B = \{3, 4\}$. Clearly, $A \cap B = \emptyset$ and $P(A \cup B) = P(\Omega) = 1$, but $P(A) + P(B) = \frac{10}{8}$. So, condition (b) does not hold and P is not a finitely additive probability measure.

Example

Prove that

If $A \subset B$, then $P(B \setminus A) = P(B) - P(A)$

Note that $B = A \cup (B \setminus A)$, where A and $B \setminus A$ are disjoint, so $P(A) + P(B \setminus A) = P(A \cup (B \setminus A)) = P(B)$. Subtracting $P(A)$ from both sides, we obtain $P(B \setminus A) = P(B) - P(A)$, as required.

COUNTABLY ADDITIVE PROBABILITY

A coin is tossed repeatedly until it lands heads up. The number of tails before the first head appears can be $0, 1, 2, \ldots$. This can be modeled by the probability space $\Omega = \{0, 1, 2, \ldots\}$, where $P(\{n\}) = \frac{1}{2^{n+1}}$ is the probability of obtaining n tails before the first head

Suppose you have made a bet that n will be even. How likely is it to happen?

Let $\mathcal{F}$ be the σ-field consisting of all subsets of Ω, so $\{n \text{ is even}\} \in \mathcal{F}$ is an event. Because

$$\{n \text{ is even}\} = \{0\} \cup \{2\} \cup \{4\} \cup \cdots ,$$

it is natural to take

$$P(\{n \text{ is even}\}) = \frac{1}{2} + \frac{1}{2^3} + \frac{1}{2^5} + \cdots = \frac{2}{3}$$

COUNTABLY ADDITIVE PROBABILITY

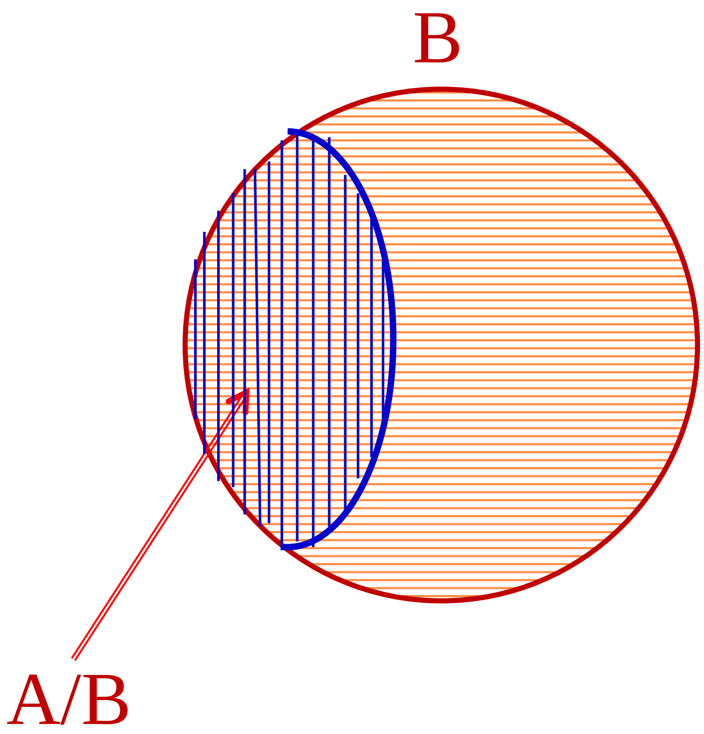
We call a function $P : \mathcal{F} \to [0,1]$ a *probability measure* if

(a) $P(\Omega) = 1$;

(b) (*countable additivity*) for any sequence $A_1, A_2, \ldots \in \mathcal{F}$ of pairwise disjoint sets (that is, $A_i \cap A_j = \emptyset$ if $i \neq j$)

$$P\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} P(A_i).$$

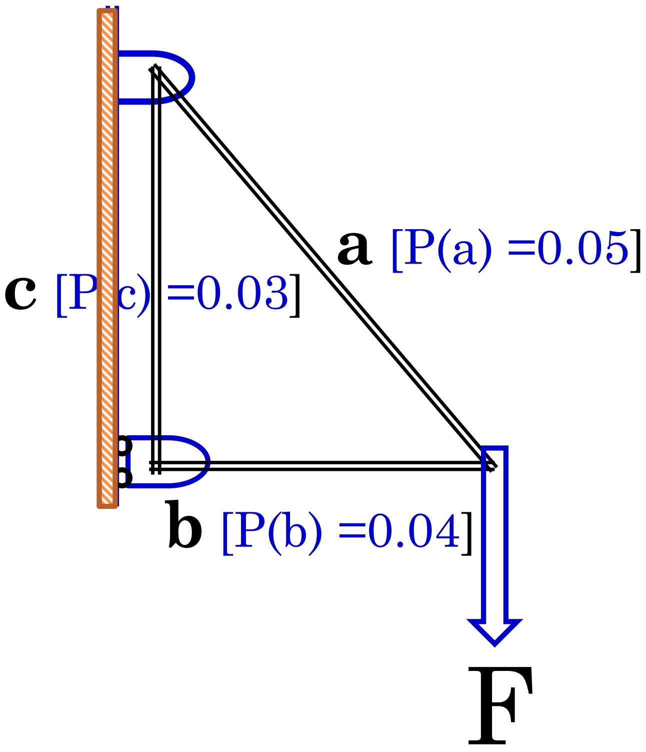
Conditional events



- Event B occurs first
- A occurs given that B has already occurred

$$P(A/B) = ?$$
$$= P(A \cap B) / P(B)$$

Example



Find the probability of failure of the truss ?

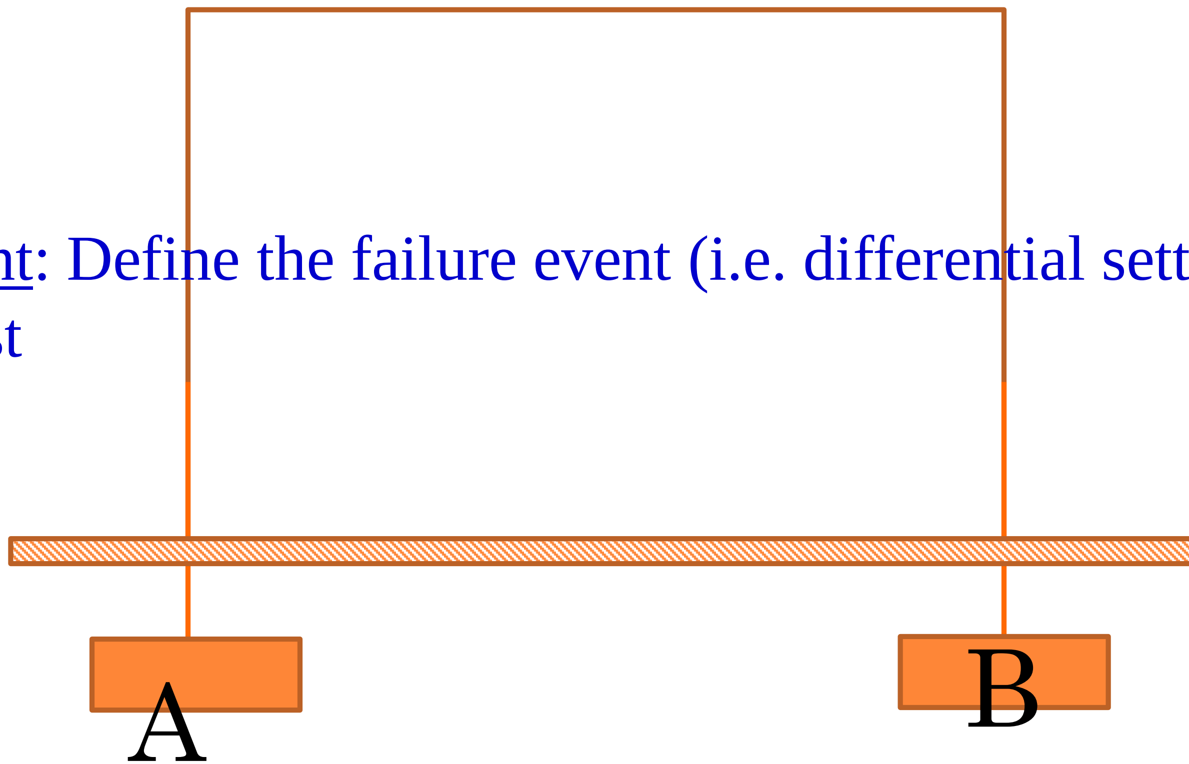
Assume: The failures of each of the members are mutually independent

Hint: Define the failure event first

Example

- Prob of settlement of each footing = 0.1
- Prob of settlement of each footing given the other one has settled = 0.8

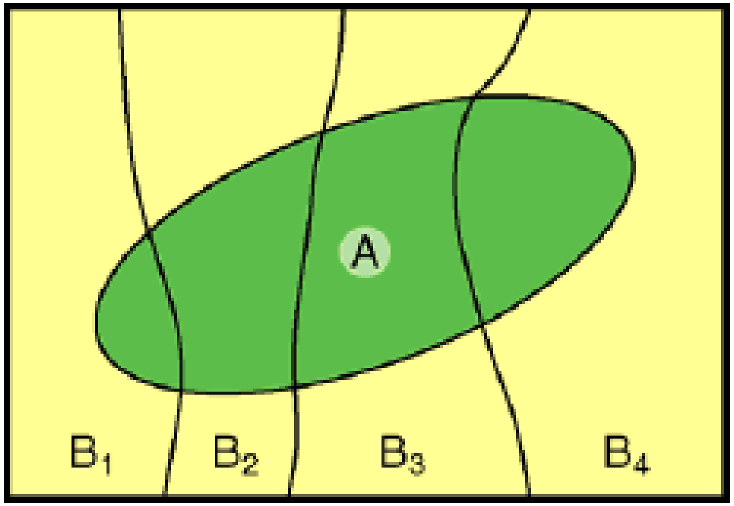
Find the probability of differential settlement



Hint: Define the failure event (i.e. differential settlement) first

Theorem of total probability

- Sometimes probability of an event A cannot be assigned directly but can be assigned conditionally for a number of other events B_i
- B_i must be mutually exclusive and collectively exhaustive



$$P(A) = P(A \mid B_1)\cdot P(B_1) + P(A \mid B_2)\cdot P(B_2) + \ldots + P(A \mid B_n)\cdot P(B_n)$$
$$= \sum_{i=1}^{n} P(A \mid B_i)\cdot P(B_i)$$

Example

There is a possibility of rain or snow tomorrow but not both. The probability of rain is 40 % while the probability of snow is 60%. If it rains then the probability that I'll be late for my work is 20%. If it snows, however, the probability of being late for my work increases to 60%.

What is the probability that I will be late for work tomorrow?

Solution:

- Let R = event that it will rain tomorrow
  S = event that it will snow tomorrow
  L = event that I will be late for my work
- We are given
  $P(R) = 0.40$ and $P(S) = 0.60$
  $P(L \mid R) = 0.20$ and $P(L \mid S) = 0.60$

Using the rule of total probability

$$P(L) = P(L \mid R) \cdot P(R) + P(L \mid S) \cdot P(S)$$
$$= (0.20)(0.40) + (0.60)(0.60) = 0.44$$

Therefore, there is a 44 % chance that I will be late for work tomorrow...

CE 513: STATISTICAL METHODS IN CIVIL ENGINEERING

Lecture-4: Random Variable

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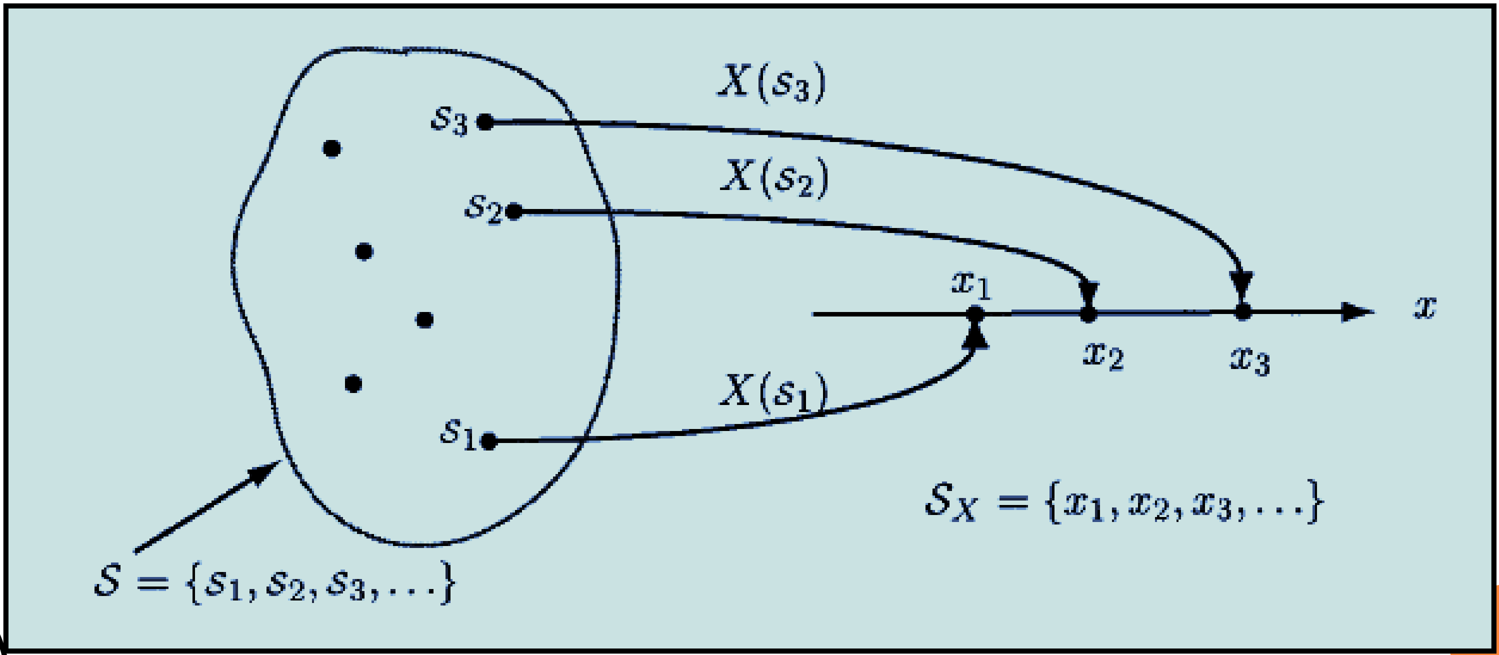
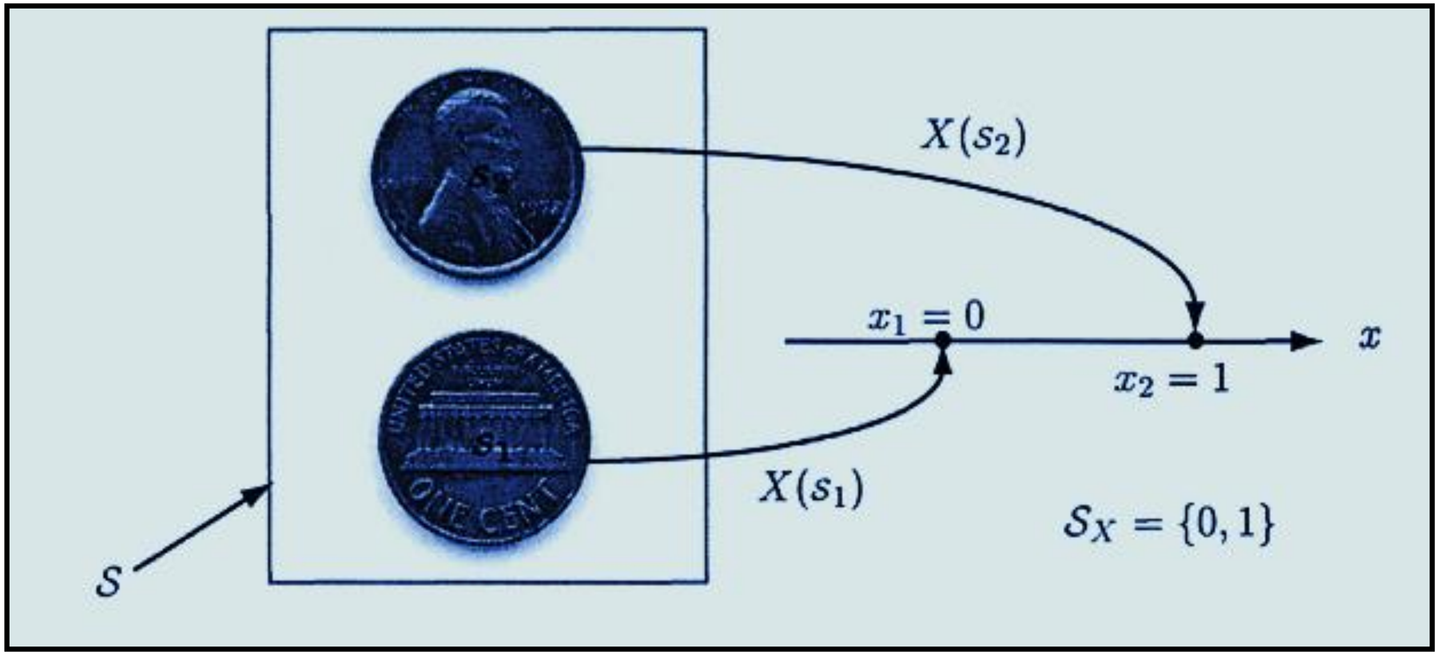
Random variable

Random Variable (RV): A finite single valued function that maps the set of all experimental outcomes in sample space S into the set of real numbers *R*, is said to be a RV



A random variable does *not* return a probability

Example: a coin toss



Random variable

- Random variable is a function X that assigns a rule of correspondence for every point ξ in the sample space S, a unique value X(ξ) on the real line $\mathcal{R}$ called the range

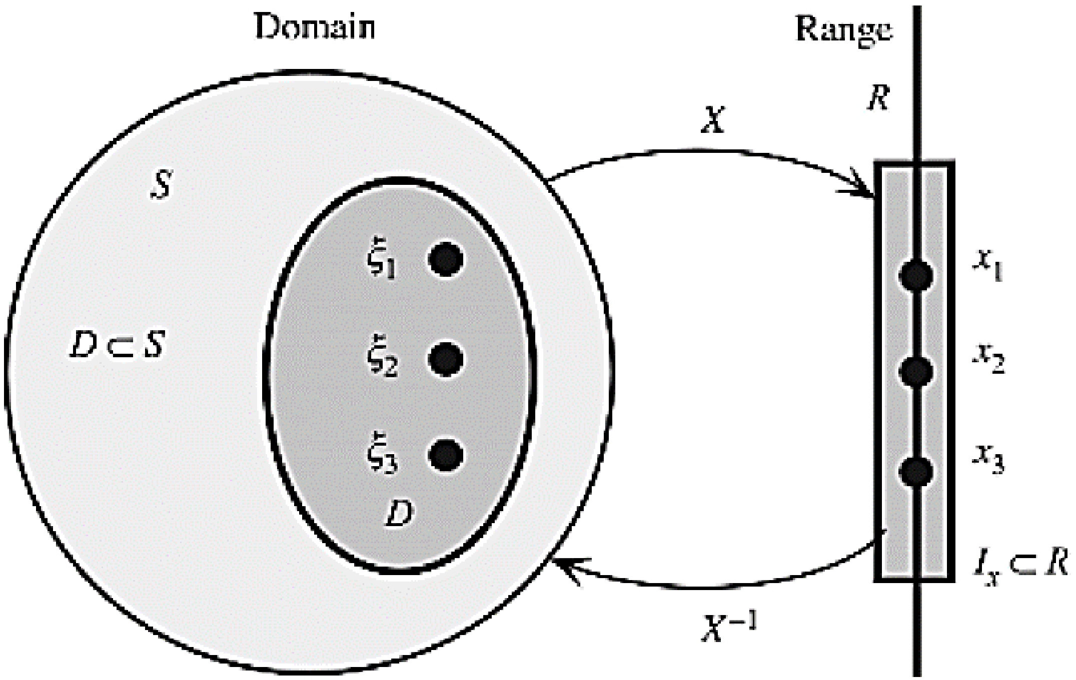
- Let Σ be the sigma field associated with the sample space and Σ_X be the sigma field associated with the real line

- The RV X *induces a probability measure* P_X in R, and hence X is a mapping of the probability space {S,Σ,P} to the probability space $\{\mathcal{R}, \Sigma_X, P_X\}$ as shown below:

$$X: \{S, \Sigma, P\} \longrightarrow \{\mathcal{R}, \Sigma_X, P_X\}$$

Random variable

$$X: \{S, \Sigma, P\} \longrightarrow \{\mathcal{R}, \Sigma_X, P_X\}$$



What sort of Sigma field is Σ_X

Discrete Random Variable

- Discrete random variables are generally used to describe events that are counted, for example: no of cars crossing the intersection
- Discrete random variables are expressed using **integers**
- The probability content of a discrete random variable is described using the **probability mass function**(PMF) and is denoted by $p_X(x)$

Discrete Random Variable

- The **cumulative distribution function**(CDF) is defined as a function of *x*, whose value is:

  The probability that X is less than or equal to x: $\mathbf{P}(X(\xi) \leq x)$

- Because the events are **mutually exclusive**(i.e. X can only assume one value at a time) the CDF is obtained simply by **adding** the discrete probabilities as

$$F_X(x) = p_X(0) + p_X(1) + \cdots + p_X(x)$$

Example: PMF

Consider the problem of three nuclear reactors.

Assume that a reactor will be active and operating 90% of the time. What is the probability that at-least two reactors are operating at a given time?

Example: PMF

Let

X = no of reactors operational at any given time

A = event that a reactor is active

O = event that a reactor is offline for service

Also let

0 = event that all reactors are offline

1 = event that 1 reactor is active and 2 are offline

2 = event that 2 reactors are active and 1 is offline

3 = event that all 3 reactors are active

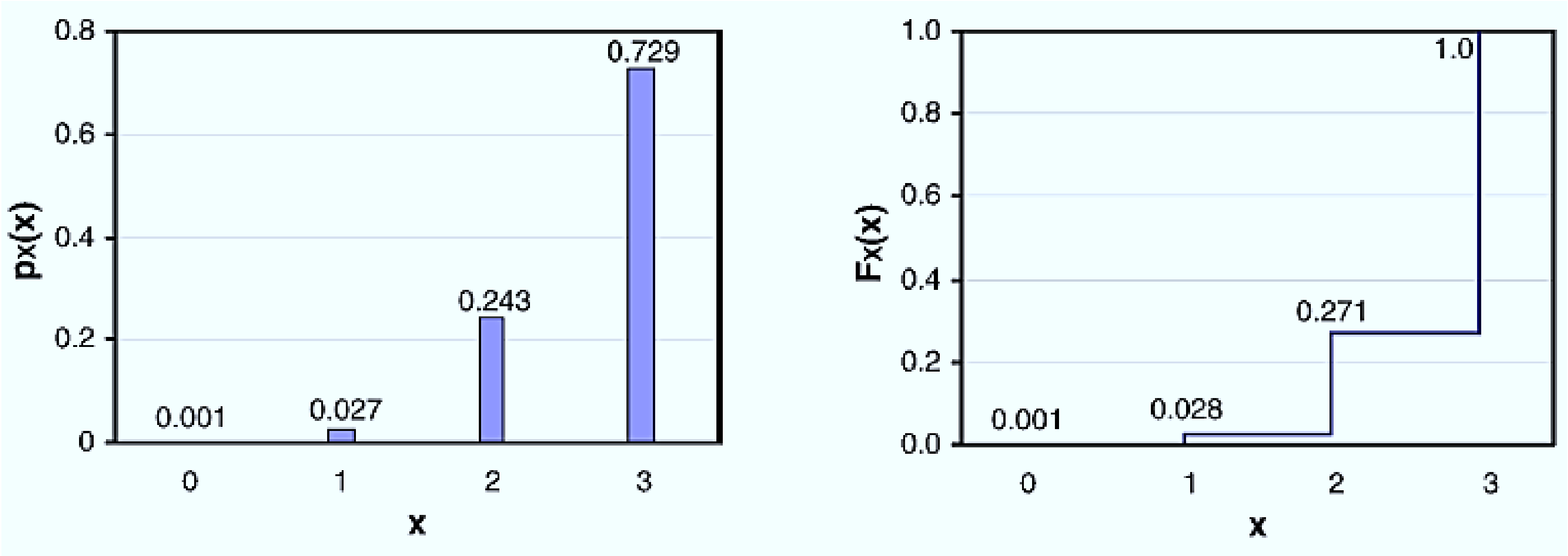
Example: PMF

We are given: $P(A) = 0.9$, $P(O) = 0.1$

Assuming the operation of the reactors is statistically independent, we can construct the PMF for the random variable X as

$p_X(0) = P(X = 0) = (0.1)(0.1)(0.1) = 0.001$
$p_X(1) = P(X = 1) = 3[(0.9)(0.1)(0.1)] = 0.027$
$p_X(2) = P(X = 2) = 3[(0.9)(0.9)(0.1)] = 0.243$
$p_X(3) = P(X = 3) = (0.9)(0.9)(0.9) = 0.729$

Example: PMF



Therefore, the probability that **at least** two reactors are operating is given by $X \geq 2$which is computed as

$$P(X \geq 2) = 1 - P(X < 2) = 1 - [p_X(0) + p_X(1)]$$
$$= 1 - [(0.001) + (0.027)] = \mathbf{0.972}$$

Properties of RV

A discrete random variable X can take m possible values X = $\{x_1, x_2, \ldots, x_m\}$ is the sample space

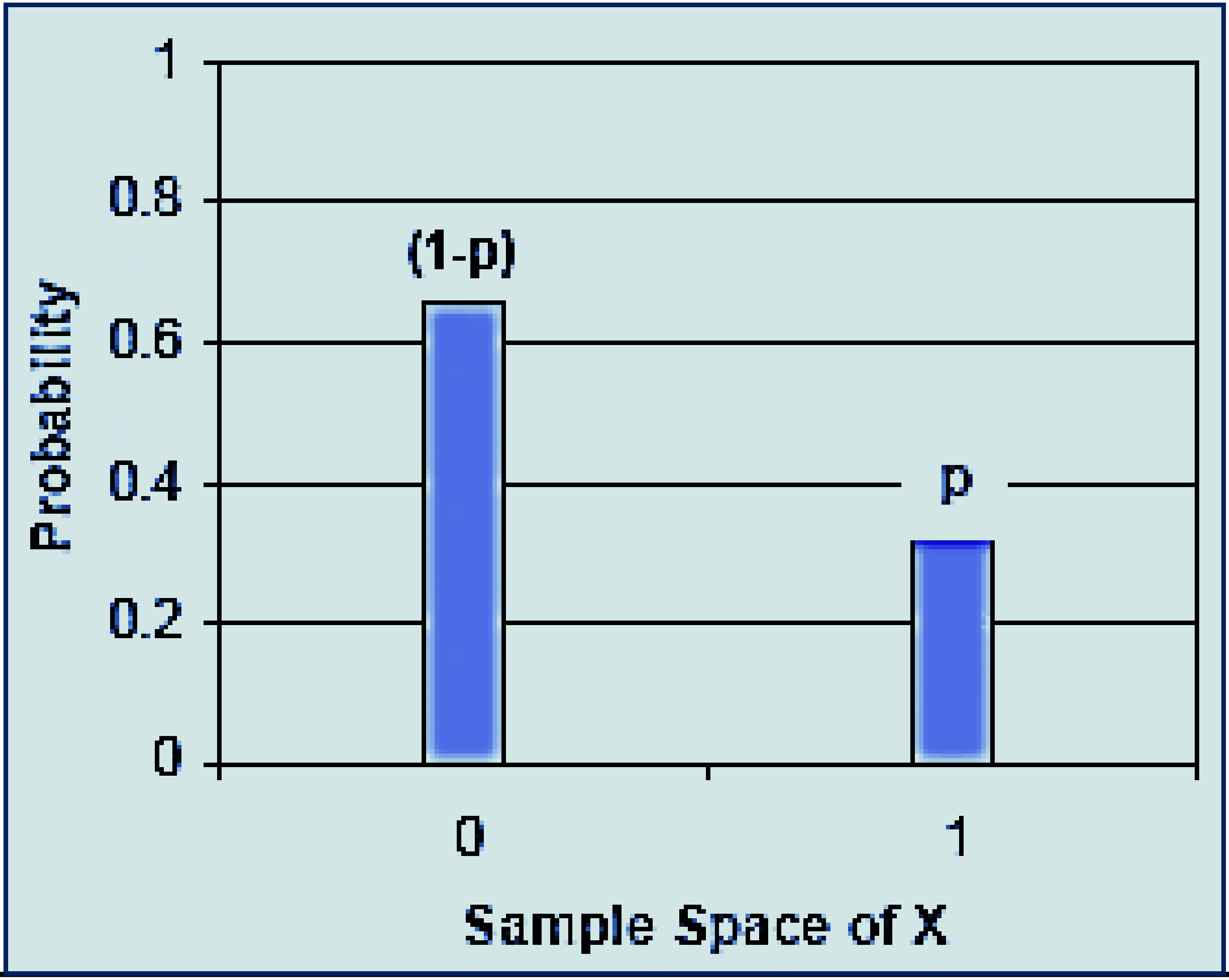
Rolling a die, X= {1, 2, 3, 4, 5, 6}

- $P(x_k)$ = Probability of RV X taking a k^{th} value (= x_k)
- Expected Value or Mean = $\mu = \sum_{k=1}^{m} P(x_k) x_k$
- Variance of X = $\sigma^2 = \sum_{k=1}^{m} P(x_k)(x_k - \mu)^2$

Bernoulli trials

- Bernoulli random variable:
    - Takes only two values, X ≡ {0, 1}
- Occurrence of an event (i.e., X = 1) with probability = *p*
- No occurrence of event (i.e., X = 0) with probability = (1-*p*)

Bernoulli trials



Bernoulli trials example

- Suppose a system has 4 standby or backup units
  The probability of failure of each unit is ***p*** *per year*
- What is the probability that 1 unit will fail in the next year ?

Unit No. Sequence	1	2	3	4	Probability
1	**F**	S	S	S	$p(1-p)^3$
2	S	**F**	S	S	$(1-p)p(1-p)^2$
3	S	S	**F**	S	$(1-p)^2p(1-p)$
4	S	S	S	**F**	$(1-p)^3p$
				Total:	**4** $p(1-p)^3$

F = Fail; **S = Safe**

Binomial Distribution

- Suppose, the distribution of the number of failures X in a group of 4 machines is a RV
- The RV follows binomial distribution

$$P(X = k) = {}^4C_k p^k (1-p)^{(4-k)}$$

$$P(X=0) = {}^4C_0 (=1)(1-p)^4$$
$$P(X=1) = {}^4C_1 (=4) p(1-p)^3$$
$$P(X=2) = {}^4C_2 (=6) p^2 (1-p)^2$$
$$P(X=3) = {}^4C_3 (=4) p^3 (1-p)^1$$
$$P(X=4) = {}^4C_4 (=1) p^4$$

Binomial Distribution

The number of trials (occurrence of transients or accidents = *m*)

- The number of failures in *m* trials = *X*, a RV ($X \leq m$)
- Probability of failure per transient/accident = p
- Binomial distribution (**Prob of exactly k occurrences in m trials**)

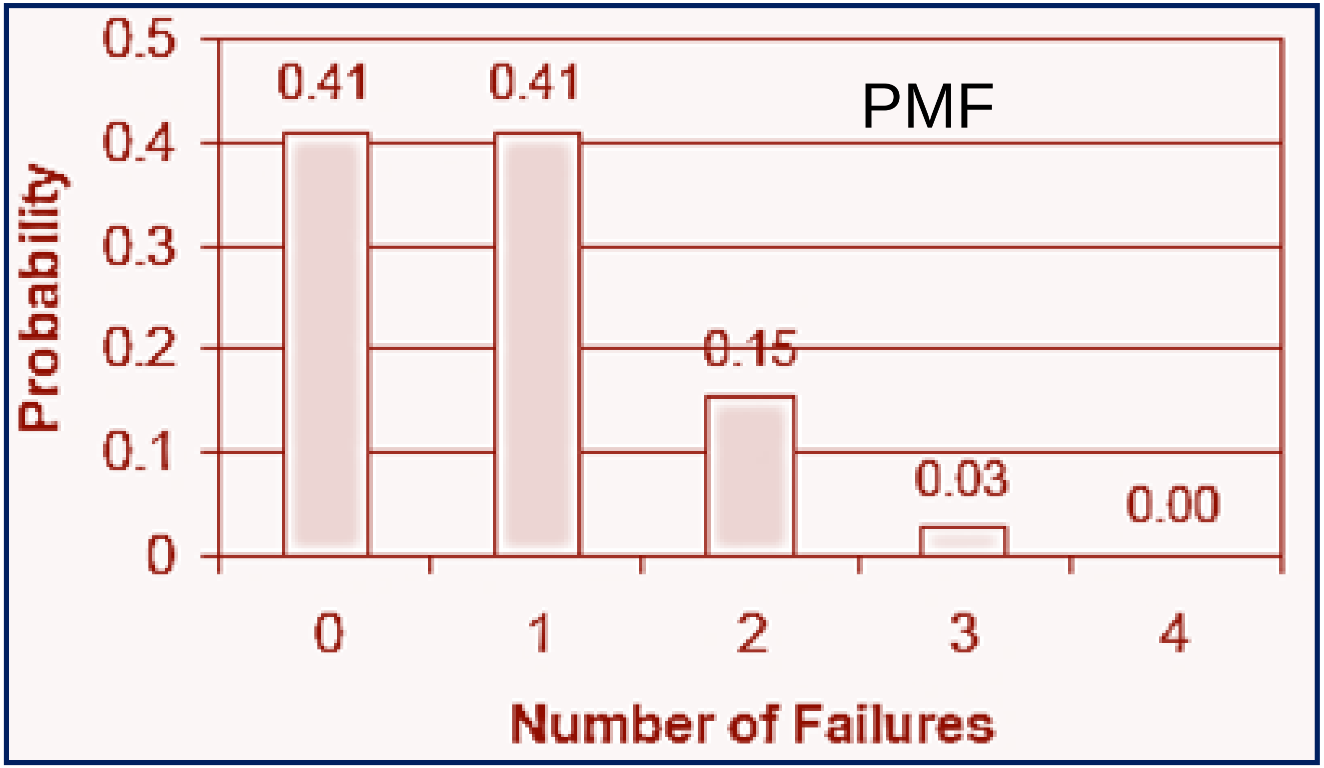
$$\boldsymbol{P(X=k) = {}^{m}C_{k}\, p^{k}(1-p)^{(m-k)}}$$

$k = 1, 2, 3, \ldots m$

- Distribution parameters are = ***m*** and ***p***

Binomial Distribution

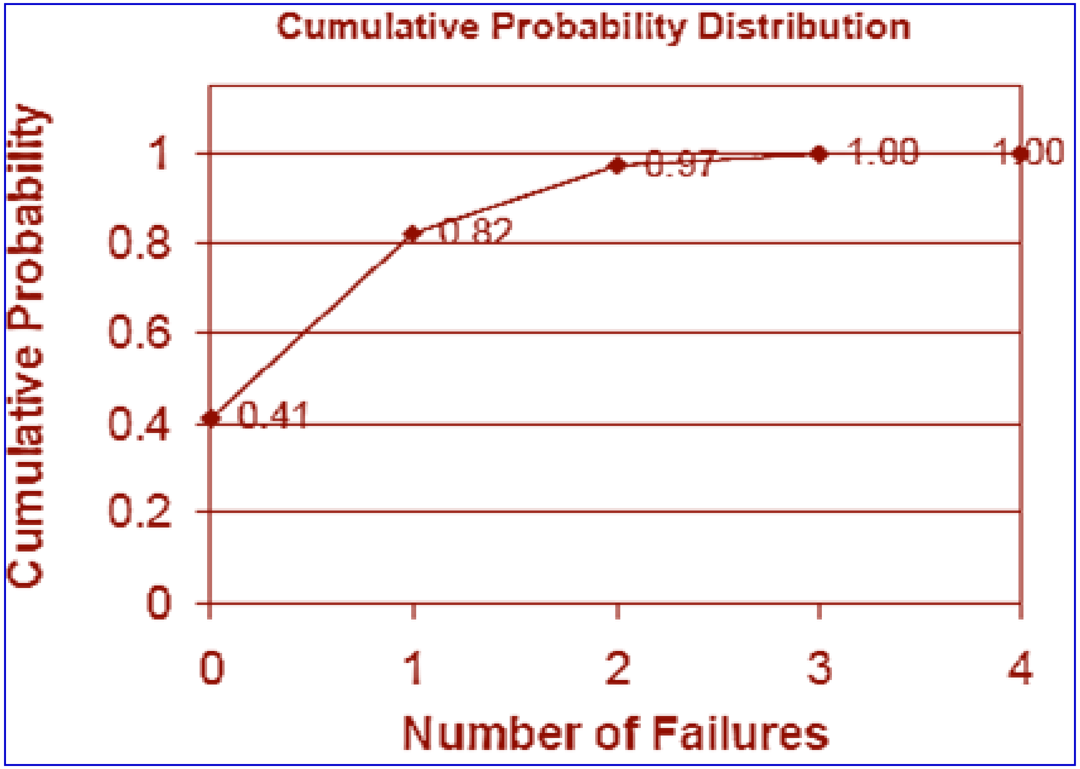
- Parameters: m = 4 machines and probability of failure p = 0.1
- The distribution of number of failures



Binomial Distribution

- What is the probability that there will be 2 or less failures? (Cumulative probability up to 2)

✓ Answer = P(X=0) + P(X=1) + P(X=2) = 0.97



Poisson Distribution

- Binomial distribution converges to the Poisson distribution
  - When probability of failure $p \rightarrow 0$ (very small)
  - And the population of component $\boldsymbol{m} \rightarrow \infty$(very large)
  - Such that $\boldsymbol{mp} \rightarrow \boldsymbol{\mu}$, constant called mean number of failures
- Poisson distribution gives the distribution of the number of failures (N)

$$P_N[k] = \frac{\mu^k e^{-\mu}}{k!} \quad (k = 0, 1, \ldots \infty)$$

Example: Poisson distribution

- Probability of failure of a component

  $p = 0.0025$ per year

- The number of components in service

  $m = 1000$

- Mean number of failures

  $\mu = m\,p = 2.5$ failures per year

$$P_N[k] = \frac{\mu^k e^{-\mu}}{k!}$$