CE 513: Statistical Methods in Civil Engineering

Lecture- 11: Random process

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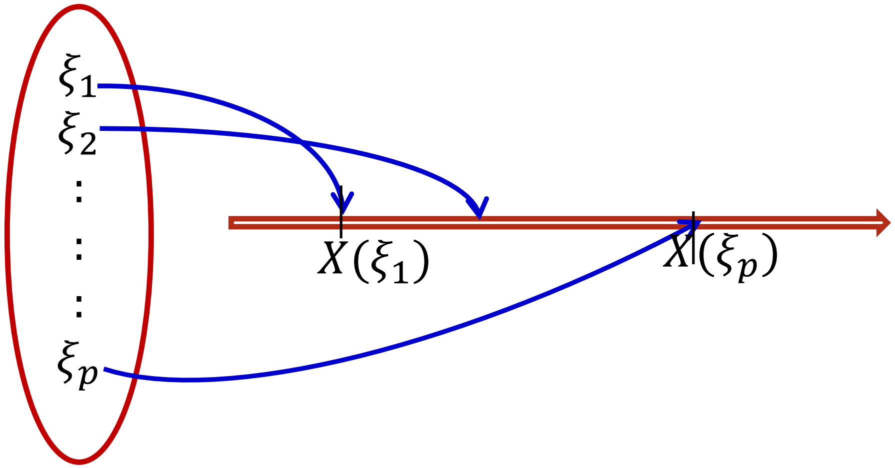
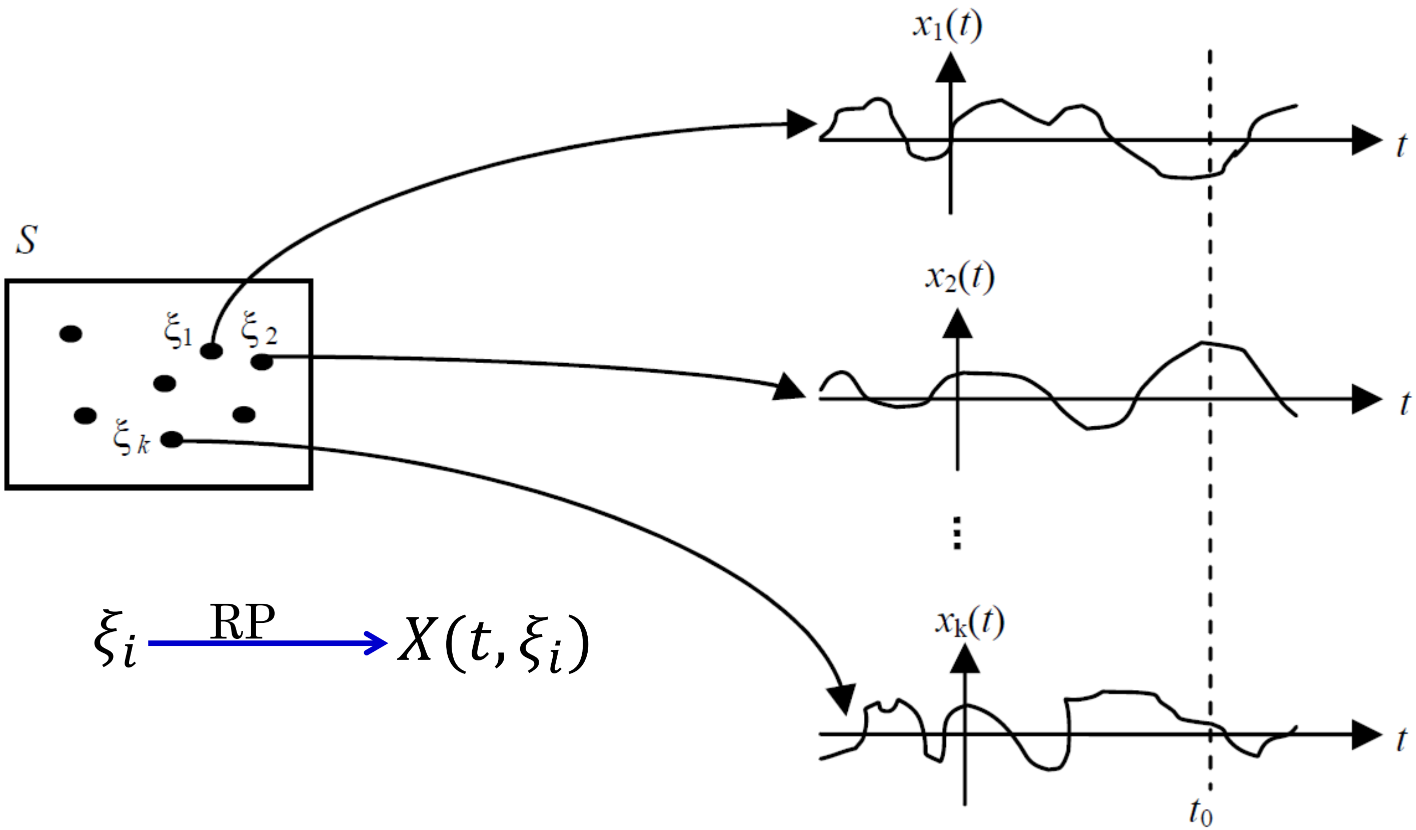
Recall: Random Variable Def



$$\xi_i \xrightarrow{\text{RV}} X(\xi_i)$$

Random process



$$\xi_i \xrightarrow{\text{RP}} X(t, \xi_i)$$

Random process

$$\xi_i \xrightarrow{\text{RP}} X(t, \xi_i)$$

A random process is a function denoted by $X(t, \xi)$

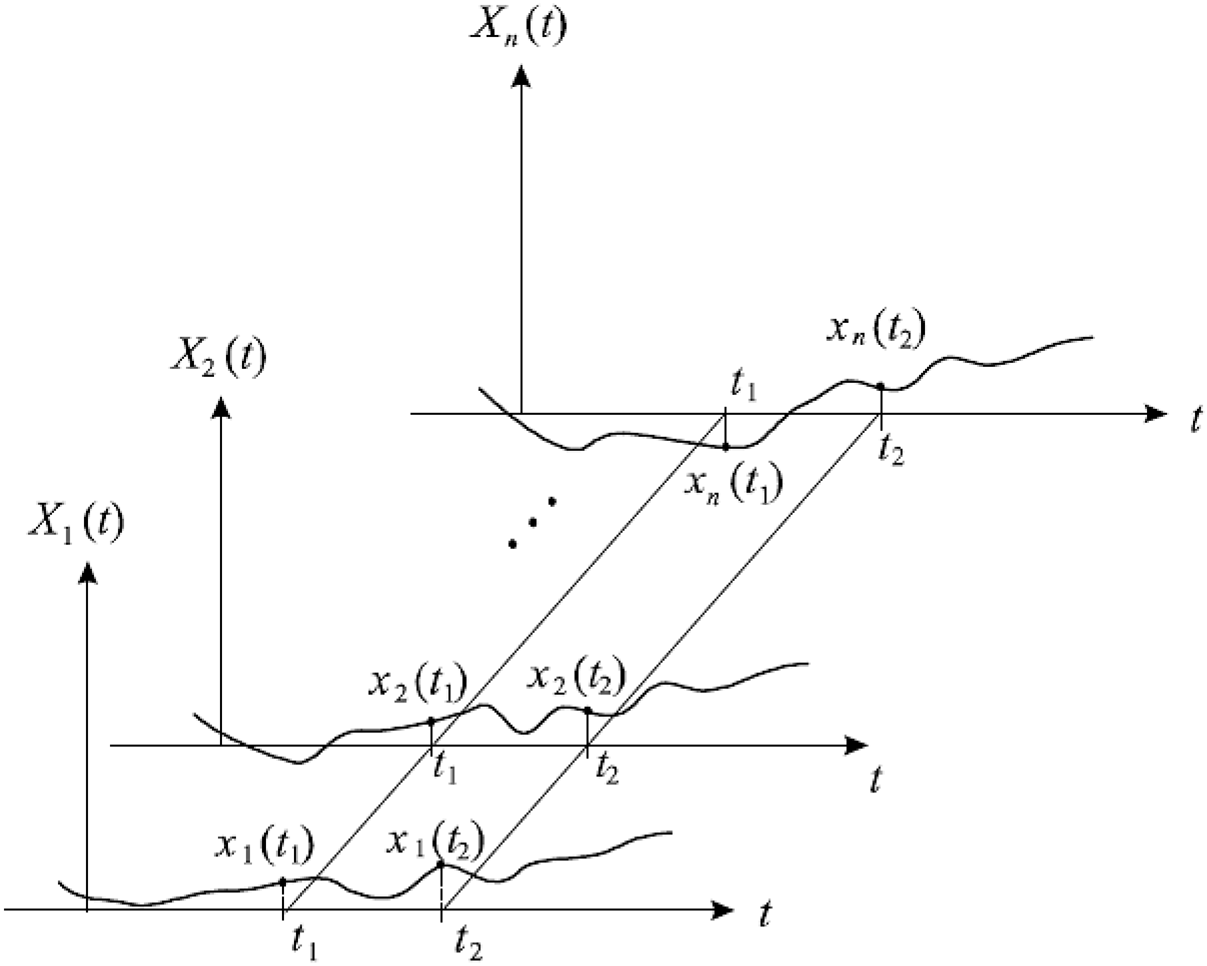
$(a)$for a fixed value of t, $X(t, \xi)$ is a random variable,

(b) for a fixed value of ξ, $X(t, \xi)$ is a function of time (a realization),

(c) for fixed values of t and ξ, $X(t, \xi)$ is a number, and

(d) for varying t and ξ, $X(t, \xi)$ is collection of time histories (ensemble)

Random process

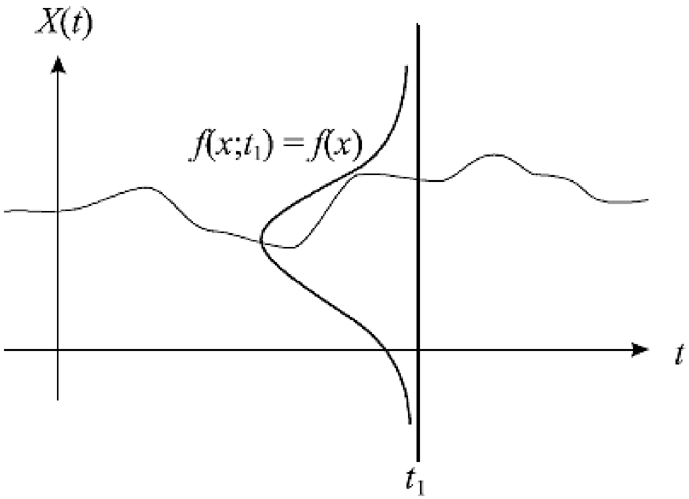


Random process

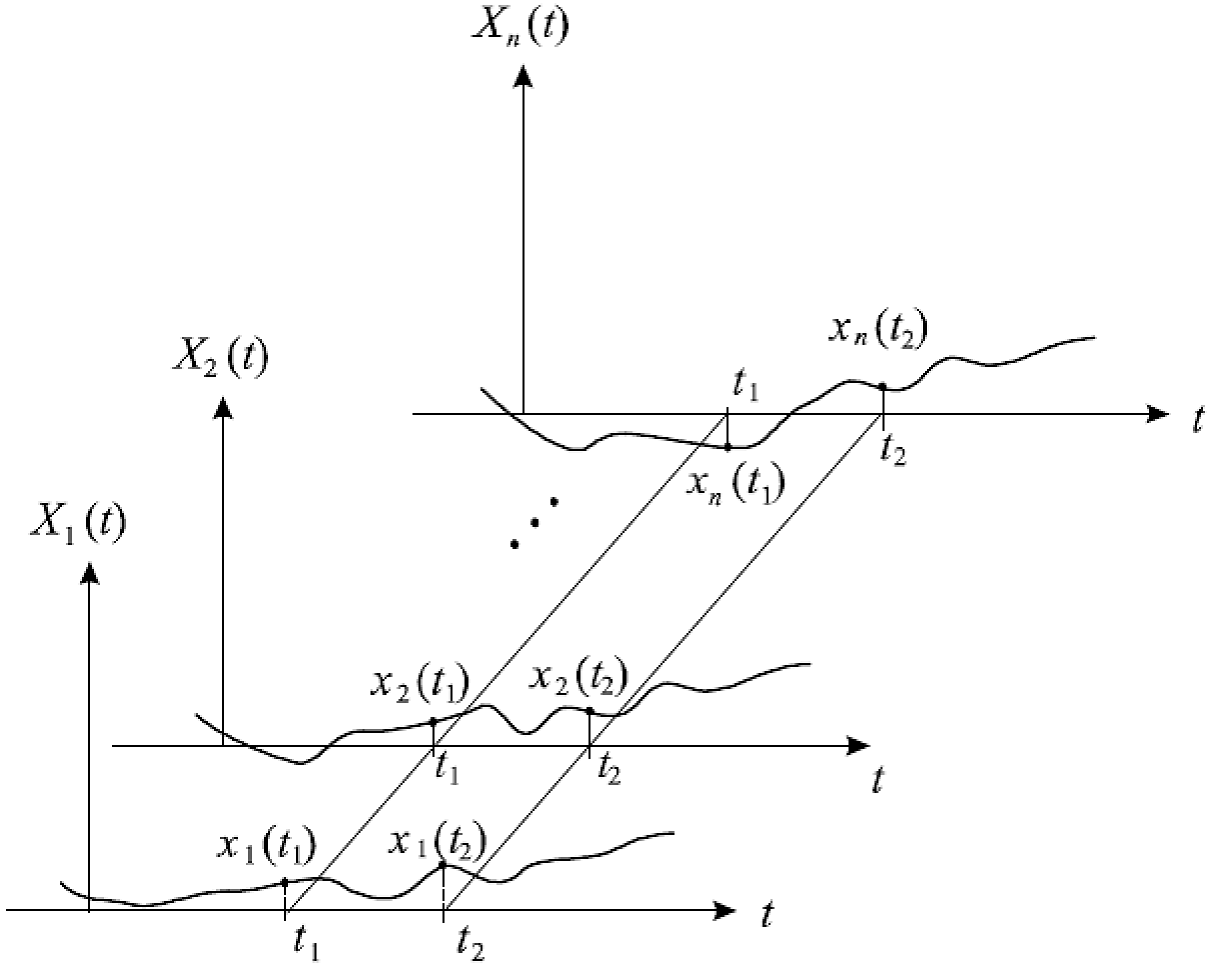
First-order distribution (for a particular value of t)

$$F_X(x;t) = P[X(t_0) \le x]$$

First-order density function $f_X(x;t) = \frac{d}{dx} F_X(x;t)$



2nd Order Averages



2^{nd} Order Averages

2^{nd} order distribution

$$F_X(x_1, x_2; t_1, t_2) = P\left[X(t_1) \le x_1 \text{ and } X(t_2) \le x_2\right]$$

2^{nd} order density function

$$f_X(x_1, x_2; t_1, t_2) = \frac{\partial^2}{\partial x_1 \partial x_2} F_X(x_1, x_2; t_1, t_2)$$

Expectations

Ensemble Average

The *mean* of $X(t)$ is defined by

$$\mu_X(t) = E[X(t)]$$

$X(t)$ is treated as a random variable for a fixed value of t.

Autocorrelation

$$R_{xx}(t_1, t_2) = E\left[X(t_1)X(t_2)\right] = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x_1 x_2 f_{X_1 X_2}(x_1, x_2; t_1, t_2)\, dx_1 dx_2$$

Autocovariance

$$\begin{aligned} K_X(t, s) = \operatorname{Cov}[X(t), X(s)] &= E\{[X(t) - \mu_X(t)][X(s) - \mu_X(s)]\} \\ &= R_X(t, s) - \mu_X(t)\mu_X(s) \end{aligned}$$

Random process

The random process $X(t)$ is given by

$$X(t) = A\cos(\omega t - \Phi),$$

where A and Φ are random variables with the probability density function,

$$f_{A\Phi}(a,\phi) = \frac{1}{2\pi}\left(1 + (3a-1)\cos\phi\right),$$
$$\text{for} \quad 0 \le \phi \le 2\pi$$
$$\text{and} \quad 0 \le a \le 1.$$

Derive (a) μ_X, (b) σ_X^2, and (c) $R_{XX}(t_1, t_2)$.

Random process

(a) The mean can be found by taking the expected value of $X(t)$ or

$$E\{X(t)\} = E\{A\cos(\omega t - \Phi)\},$$

which can be expanded to

$$E\{X(t)\} = E\{A(\cos\omega t\cos\Phi + \sin\omega t\sin\Phi)\}.$$

Since only A and ϕ are random, $\cos\omega t$ and $\sin\omega t$ can be taken out of the expectation so that

$$E\{X(t)\} = \cos\omega t E\{A\cos\Phi\} + \sin\omega t E\{A\sin\Phi\},$$

Random process

$$E\{X(t)\} = \cos\omega t E\{A\cos\Phi\} + \sin\omega t E\{A\sin\Phi\},$$

where

$$\begin{aligned} E\{A\cos\Phi\} &= \int_0^1\int_0^{2\pi} a\cos\phi f_{A\Phi}(a,\phi)\,d\phi da \\ &= \frac{1}{4} \\ E\{A\sin\Phi\} &= \int_0^1\int_0^{2\pi} a\sin\phi f_{A\Phi}(a,\phi)\,d\phi da \\ &= 0. \end{aligned}$$

Then,

$$E\{X(t)\} = \frac{1}{4}\cos\omega t,$$

which means that $X(t)$ is a nonstationary random process.

Random process

(b) The variance can be found using

$$\sigma_X^2 = E\left\{(X(t) - \mu_X)^2\right\} = E\left\{X^2\right\} - \mu_X^2.$$

The root mean square $E\left\{X^2\right\}$ is given by

$$\begin{aligned} E\left\{X^2\right\} &= E\left\{A^2(\cos\omega t\cos\Phi + \sin\omega t\sin\Phi)^2\right\} \\ &= E\{A^2\cos^2\omega t\cos^2\Phi + A^2\sin^2\omega t\sin^2\Phi \\ &\quad +2A^2\cos\omega t\cos\Phi\sin\omega t\sin\Phi\} \\ &= \cos^2\omega t E\left\{A^2\cos^2\Phi\right\} + \sin^2\omega t E\left\{A^2\sin^2\Phi\right\} \\ &\quad +2\cos\omega t\sin\omega t E\left\{A^2\cos\Phi\sin\Phi\right\}. \end{aligned}$$

Random process

(b) The variance can be found

Each term in the previous equation can be evaluated as follows

$$
\begin{aligned}
E\left\{A^2 \cos^2 \Phi\right\} &= \frac{1}{6} \\
E\left\{A^2 \sin^2 \Phi\right\} &= \frac{1}{6} \\
E\left\{A^2 \cos \Phi \sin \Phi\right\} &= 0.
\end{aligned}
$$

Then,

$$E\left\{X^2\right\} = \frac{1}{6}$$

and the variance equals

$$
\begin{aligned}
\sigma_X^2 &= R_{XX}(t,t) - \mu_X^2 \\
&= E\left\{X^2\right\} - \mu_X^2 = \frac{1}{6} - \frac{1}{16} \cos \omega^2 t.
\end{aligned}
$$

Random process

(c) The autocorrelation function $R_{XX}(t_1, t_2)$, by definition, is given by

$$\begin{aligned} R_{XX}(t_1, t_2) &= E\left\{A^2 \cos(\omega t_1 - \Phi)\cos(\omega t_2 - \Phi)\right\} \\ &= \cos\omega t_1 \cos\omega t_2 E\{A^2 \cos^2\Phi\} + \sin\omega(t_1 + t_2) E\left\{A^2 \cos\Phi \sin\Phi\right\} \\ &\quad + \sin\omega t_1 \sin\omega t_2 E\{A^2 \sin^2\Phi\} \\ &= \frac{1}{6}\cos\omega(t_1 - t_2). \end{aligned}$$

Autocorrelation: example

Consider the random process $X(t)$ $\quad X(t) = Y \cos \omega t \quad t \geq 0$

where ω is a constant and Y is a uniform r.v. over $(0, 1)$.

(*a*) Find $E[X(t)]$.

(*b*) Find the autocorrelation function $R_x(t, s)$ of $X(t)$.

(*c*) Find the autocovariance function $K_x(t, s)$ of $X(t)$

Autocorrelation: example

(a) $E(Y) = \frac{1}{2}$ and $E(Y^2) = \frac{1}{3}$. Thus,

$$E[X(t)] = E(Y \cos \omega t) = E(Y) \cos \omega t = \frac{1}{2} \cos \omega t$$

(b)
$$R_X(t, s) = E[X(t)X(s)] = E(Y^2 \cos \omega t \cos \omega s)$$
$$= E(Y^2) \cos \omega t \cos \omega s = \frac{1}{3} \cos \omega t \cos \omega s$$

(c)
$$K_X(t, s) = R_X(t, s) - E[X(t)]E[X(s)]$$
$$= \frac{1}{3} \cos \omega t \cos \omega s - \frac{1}{4} \cos \omega t \cos \omega s$$
$$= \frac{1}{12} \cos \omega t \cos \omega s$$

Classification of stochastic process

Strictly stationary

A random process $\{X(t), t \in T\}$ is said to be *stationary* or *strict-sense stationary* if, for all n and for every set of time instants $(t_i \in T, i = 1, 2, \ldots, n\}$,

$$F_X(x_1, \ldots, x_n; t_1, \ldots, t_n) = F_X(x_1, \ldots, x_n; t_1 + \tau, \ldots, t_n + \tau)$$

Thus both first order and second order distributions are independent of t

$$F_X(x; t) = F_X(x; t + \tau) = F_X(x)$$
$$f_X(x; t) = f_X(x)$$
$$\mu_X(t) = E[X(t)] = \mu$$
$$\text{Var}[X(t)] = \sigma^2$$

$$F_X(x_1, x_2; t_1, t_2) = F_X(x_1, x_2; t_2 - t_1)$$
$$f_X(x_1, x_2; t_1, t_2) = f_X(x_1, x_2; t_2 - t_1)$$

Wide sense stationary

If stationary condition of a random process X(t) does not hold for all n but holds for n ≤ k, then we say that the process X(t)is stationary to order k.

If X(t) is stationary to order 2, then X(t) is said to be wide-sense stationary (WSS) or weak stationary.

1. $E[X(t)] = \mu$ (constant)
2. $R_X(t, s) = E[X(t)X(s)] = R_X(|s - t|)$

Wide sense stationary

- Stationarity of a random process
  is analogous to
  steady state in vibration problems

- One or more of the properties of random process becomes independent of time

- Strong sense stationarity (SSS) : defined with respect to pdf-s

- Wide sense stationarity (WSS) : defined with respect to moments

Stationary SS: Few Theorems

1. If a random process which is stationary to order **n** is also stationary to all orders lower than **n.**

2. If $\{X(t), t \in T\}$ is a strict-sense stationary random process, then it is also WSS.

3. If a random process X(t) is WSS, then it must also be covariance stationary

SSS: Example

Consider a random process $X(t)$ defined by

$$X(t) = U \cos \omega t + V \sin \omega t \qquad -\infty < t < \infty$$

where ω is constant and U and V are r.v.'s.

(*a*) Show that the condition

$$E(U) = E(V) = 0$$

is necessary for $X(t)$ to be stationary.

(*b*) Show that $X(t)$ is WSS if and only if U and V are uncorrelated with equal variance; that is,

$$E(UV) = 0 \qquad E(U^2) = E(V^2) = \sigma^2$$

(*a*)

$$\mu_X(t) = E[X(t)] = E(U)\cos\omega t + E(V)\sin\omega t$$

must be independent of t for $X(t)$ to be stationary.

This is possible only if $\mu_x(t) = 0$, that is, $E(U) = E(V) = 0$.

(*b*) If $X(t)$ is WSS, then

$$E[X^2(0)] = E\left[X^2\left(\frac{\pi}{2\omega}\right)\right] = R_{XX}(0) = \sigma_X{}^2$$

But $X(0) = U$ and $X(\pi/2\omega) = V$; thus,

$$E(U^2) = E(V^2) = \sigma_X{}^2 = \sigma^2$$

Using the above result, we obtain

$$\begin{aligned} R_x(t, t+\tau) &= E[X(t)X(t+\tau)] \\ &= E\{(U\cos\omega t + V\sin\omega t)[U\cos\omega(t+\tau) + V\sin\omega(t+\tau)]\} \\ &= \sigma^2\cos\omega\tau + E(UV)\sin(2\omega t + \omega\tau) \end{aligned}$$

Conversely, if $E(UV) = 0$ and $E(U^2) = E(V^2) = \sigma^2$, then from the result of part (*a*) and the above result

$$\mu_x(t) = 0$$
$$R_x(t, t+\tau) = \sigma^2\cos\omega\tau = R_x(\tau)$$

$R_x(\tau)$ (WSS) examples

1) $G(t) = A\cos(\omega_0 t + \phi)$, where ϕ is uniform RV with $\phi \sim U(0, 2\pi)$. Determine the mean and the autocorrelation ?

Ans = $\frac{A^2}{2}\cos(\omega_0 \tau)$

2) $G(t) = A\cos(\omega t + \theta)$, where ω and θ are independent RVs with $\theta \sim U(0, 2\pi)$ and $\omega \sim U(\omega_1, \omega_2)$. Determine the mean and the autocorrelation ?

Ans = $\frac{A^2}{2\tau(\omega_2 - \omega_1)}[\sin\omega_2\tau - \sin\omega_1\tau]$

Autocorrelation: Properties

1. It is an even function of τ

$$R_x(\tau) = R_x(-\tau)$$

2. Bounded by its value at origin

$$|R_x(\tau)| \leq R_x(0)$$

3. $R_x(0) = E[X^2]$

4. If X is periodic $R_x(\tau)$ is also periodic

Autocorrelation: Example

A random process $Y(t)$ is given by $Y(t) = X(t)\cos(\omega t + \Phi)$, where $X(t)$, a zero mean wide-sense stationary random process with autocorrelation function $R_X(\tau) = 2e^{-2\lambda|\tau|}$ is modulating the carrier $\cos(\omega t + \Phi)$. The random variable Φ is uniformly distributed in the interval $(0, 2\pi)$, and is independent of $X(t)$. We have to find the mean, variance, and autocorrelation of $Y(t)$:

Autocorrelation: Example

Mean. The independence of $X(t)$ and Φ allows us to write

$$E[Y(t)] = E[X(t)]E[\cos(\omega t + \Phi)]$$

and with $E[X(t)] = 0$ and $E[\cos(\omega t + \Phi)] = 0$ $\quad E[Y(t)] = 0$

Variance. Since $X(t)$ and Φ are independent, the variance can be given by

$$\sigma_Y^2 = E[Y^2(t)] = E[X^2(t)\cos^2(\omega t + \Phi)] = \sigma_X^2 E[\cos^2(\omega t + \Phi)]$$

However

$$E[\cos^2(\omega t + \Phi)] = \frac{1}{2}E[1 + \cos(2\omega t + 2\Phi)] = \frac{1}{2} \quad \text{and} \quad \sigma_X^2 = C_X(0) = R_X(0) = 2$$

and hence $\sigma_Y^2 = \sigma_X^2/2 = 1$.

Autocorrelation: Example

Autocorrelation:

$$R_Y(\tau) = E[Y(t)Y(t+\tau)] = E[X(t)\cos(\omega t + \Phi)X(t+\tau)\cos(\omega t + \omega\tau + \Phi)]$$

$$= R_X(\tau)\frac{1}{2}E[\cos(\omega\tau) + \cos(2\omega t + \omega\tau + 2\Phi)]$$

$$= \frac{R_X(\tau)}{2}\cos(\omega\tau) + \frac{R_X(\tau)}{2}E[\cos(2\omega t + \omega\tau + 2\Phi)]$$

$E[\cos(2\omega t + \omega\tau + 2\Phi)] = 0$, and hence

$$R_Y(\tau) = \frac{R_X(\tau)}{2}\cos(\omega\tau) = e^{-2\lambda|\tau|}\cos(\omega\tau)$$

A graph of $R_Y(\tau)$ is shown in Fig. with $\lambda = 0.5$ and $\omega = 2\pi$.

Cross-correlation

1. Two processes X(t) and Y(t) are called jointly stationary

- if each of them are WSS individually
- $R_{xy}(t, t+\tau) = R_{xy}(\tau)$
  $R_{yx}(t, t+\tau) = R_{yx}(\tau)$

2. $R_{xy}(\tau)$ and $R_{yx}(\tau)$ are mirror images of each other $R_{xy}(\tau) = R_{yx}(-\tau)$

White-Noise

Consider a time function that switches between two values $+a$ and $-a$ as shown
The crossing times t_i are random and we assume that it is modelled as a Poisson process with a rate parameter λ. Then, the probability of k crossings in time τ is

$$p_k = \frac{e^{-\lambda|\tau|}(\lambda|\tau|)^k}{k!}$$

where λ is the number of crossings per unit time.

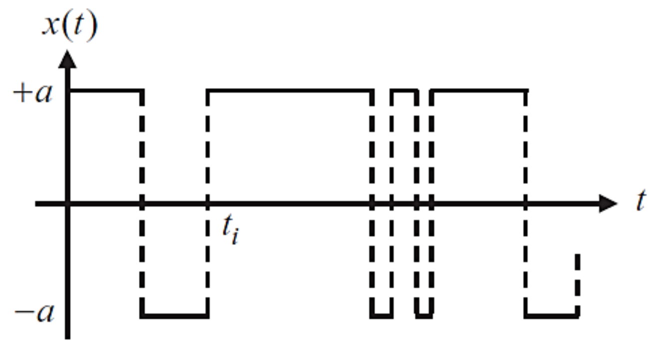
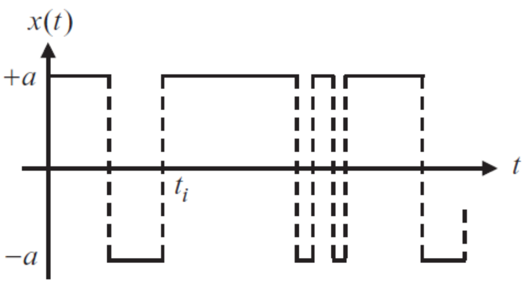


Figure Asynchronous random telegraph signal

White-Noise



If we assume that the process is in steady state, i.e. $t \to \infty$,

then $P[X(t) = a] = P[X(t) = -a] = 1/2$

So the mean value $\mu_x = E[x(t)] = 0$.

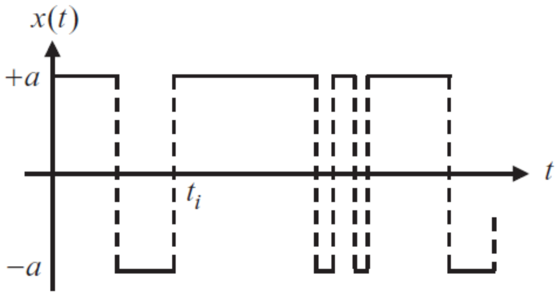
The product $x(t)x(t+\tau)$ is either a^2 or $-a^2$

it is a^2 if the number of crossings is even in time τ

$-a^2$ if the number of crossings is odd in time τ

The total probability for a^2 (i.e. an even number of crossings occurs) is $\sum_{k=0}^{\infty} p_{2k}$, and the total probability for $-a^2$ is $\sum_{k=0}^{\infty} p_{2k+1}$

White-Noise



Thus, the autocorrelation function becomes

$$R_{xx}(\tau) = E\left[x(t)x(t+\tau)\right] = \sum_{k=0}^{\infty}\left[a^2 p_{2k} - a^2 p_{2k+1}\right]$$

$$= a^2 e^{-\lambda|\tau|}\left[\sum_{k=0}^{\infty}\left(\frac{(\lambda|\tau|)^{2k}}{(2k)!} - \frac{(\lambda|\tau|)^{2k+1}}{(2k+1)!}\right)\right] = a^2 e^{-\lambda|\tau|}\left[\sum_{k=0}^{\infty}\frac{(-\lambda|\tau|)^k}{k!}\right]$$

$$= a^2 e^{-2\lambda|\tau|}$$

White-Noise

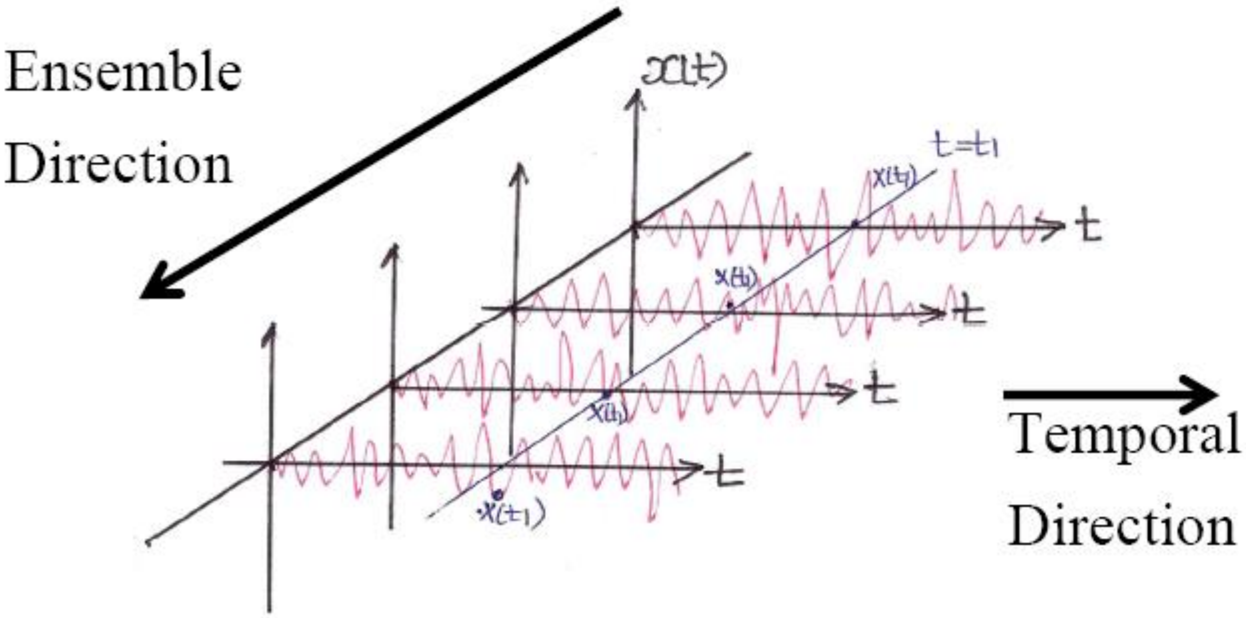
- Note that, as the parameter λ gets larger, $R_x(\tau)$ becomes narrower
- We use this to define a special WSS called White Noise
- As $\lambda \rightarrow \infty$, the process is very erratic and $R_x(\tau)$ becomes a Dirac delta function
- In order that $R_x(\tau)$ doesn't disappear completely, a becomes very large

This gives an idea of a 'completely erratic' random process whose autocorrelation (autocovariance) function is like a delta function, and the process that has this property is called *white noise*, i.e.

$$\text{Autocorrelation function of white noise: } R_{xx}(\tau) = k\delta(\tau)$$

Erogodicity

Basic idea: Equivalence of temporal and ensemble averages



Erogodicity

A random process is said to be Ergodic if it has the property that the time averages of sample functions of the process are equal to the corresponding statistical or ensemble averages.

$$E[X(t)] = \langle X(t) \rangle = \frac{1}{T}\int_{-T/2}^{T/2} x(t)dt$$

The sample autocorrelation can be calculated using the following formula

$$R_X(\tau) = \langle X(t)X(t+\tau) \rangle = \frac{1}{T}\int_{-T/2}^{T/2} x(t)x(t+\tau)\, dt$$

Erogodicity

- Consider a sample of a random process: x (1), x (2),………x (N)
- The sample mean of the sequence could be estimated as:

$$\widehat{m_x}(N) = \frac{1}{N}\sum_{n=0}^{N-1} x_n$$

- Since the sample is a realization of a random process it must have a constant ensemble mean E[X(n)]= m_x

If the sample mean $\widehat{m_x}(N)$ of a WSS converges to m_x in a *mean square sense* as N→ ∞, then the random process is said to be Ergodic in mean

$$\lim_{N\to\infty} \widehat{m_x}(N) = m_x$$

Mean Ergodic Theorem

Mean Ergodic Theorem 1. Let $x(n)$ be a WSS random process with autocovariance sequence $c_x(k)$. A necessary and sufficient condition for $x(n)$ to be ergodic in the mean is

$$\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N-1} c_x(k) = 0$$

Mean Ergodic Theorem 2. Let $x(n)$ be a WSS random process with autocovariance sequence $c_x(k)$. Sufficient conditions for $x(n)$ to be ergodic in the mean are that $c_x(0) < \infty$ and

$$\lim_{k\to\infty} c_x(k) = 0$$

Sample autocorrelation of a WSS and Ergodic process

$$r_x(k) = E[\, x(k)x(n-k)]$$

For each k, the autocorrelation is the expected value of the process: $y_k(n) = x(k)x(n-k)$

Using Ergodicity properties, the autocorrelation is finally estimated as :

$$\widehat{r_x}(k, N) = \frac{1}{N}\sum_{n=0}^{N-1} x(k)x(n-k)$$

WSS& Ergodic process: example

Coming back to the random phase sinusoid

$G(t) = A\cos(\omega_0 t + \phi)$, where ϕ is uniform RV with $\phi \sim U(0, 2\pi)$.

$$\langle X(t)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)dt = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A\cos(\omega_0 t + \phi)\, dt = 0$$

$$\langle X(t)X(t+\tau)\rangle = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} x(t)x(t+\tau)dt$$

$$= \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T} A^2\cos(\omega_0 t + \omega_0\tau + \phi)cos(\omega_0 t + \phi)dt$$

$$= \frac{A^2}{2}\cos(\omega_0\tau)$$

Applications

Noisy signals

Consider a signal buried in white-noise, i.e. y(t) = s(t) + n(t)

Assume: Noise and signal are uncorrelated and with mean = 0

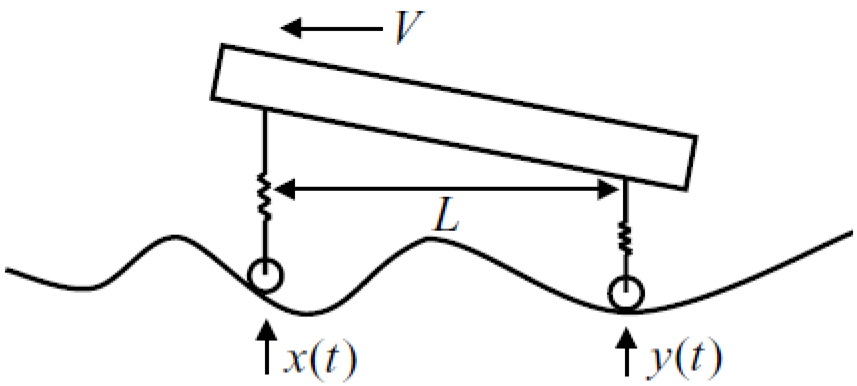
Therefore: $R_{sn}(\tau) = E[s(t)n(t+\tau)] = \mu_s \mu_n$

$$R_{yy}(\tau) = E\left[(s(t)+n(t))(s(t+\tau)+n(t+\tau))\right]$$
$$= E\left[s(t)s(t+\tau)\right] + E\left[n(t)n(t+\tau)\right] + 2\mu_s\mu_n$$

$$R_{yy}(\tau) = R_{ss}(\tau) + R_{nn}(\tau)$$

As Rnn(τ) decays very rapidly, the autocorrelation function of the signal *Rss* (τ) will dominate for larger values of τ

Application of cross-correlation



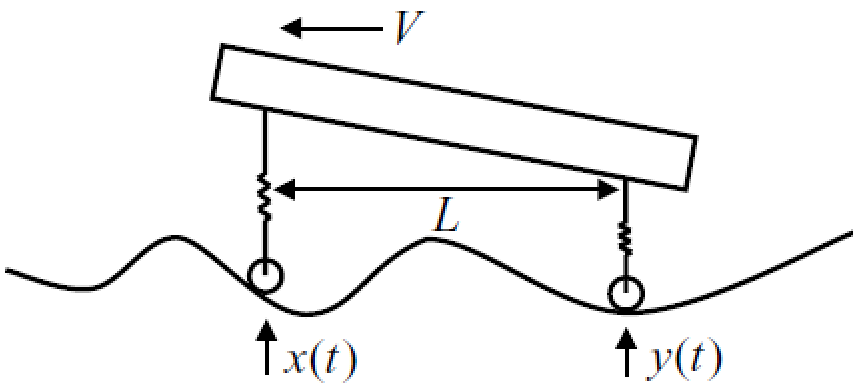
Consider a wheeled vehicle moving over rough terrain as shown in Figure.

- Let the time function (profile) experienced by the leading wheel be $x(t)$ and that by the trailing wheel be $y(t)$
- Let the autocorrelation of x (t) be $R_{xx}(\tau)$
- Assume that the vehicle moves at a constant speed V.

  Then, $y(t) = x(t - \Delta)$ where $\Delta = L/V$

$$R_{xy}(\tau) = E[x(t)y(t+\tau)] = E[x(t)x(t+\tau-\Delta)]$$
$$= R_{xx}(\tau - \Delta)$$

Application of cross-correlation



- Let x (t) and y(t) be observed in presence of white noise ($\sim N(0, \sigma^2)$)

$$x(t) = s(t) + n_x(t)$$
$$y(t) = s(t - \Delta) + n_y(t)$$

The cross-correlation function $R_{xy}(\tau)$ is (assuming zero mean values)

$$R_{xy}(\tau) = E[(s(t) + n_x(t))\left(s(t - \Delta + \tau) + n_y(t + \tau)\right)]$$
$$= E\left[s(t)s(t + \tau - \Delta)\right] = R_{ss}(\tau - \Delta)$$

MATLAB examples

- Autocorrelation of a random phase sinusoid
- Noisy signal
- Time delay problem

Independent Increment Processes

- Independent Increment Process:

- $\{X(t)$, t> 0 } is said to have independent increments when

  $X(0), X(t_1) - X(0), X(t_2) - X(t_1), \ldots \ldots \ldots \ldots, X(t_n) - X(t_{n-1})$

  are independent

- If $\{X(t)$, t> 0 } possesses independent increments and

  $X(t+h) - X(s+h)$ has the same distn as $X(t)$- $X(s)$, then process

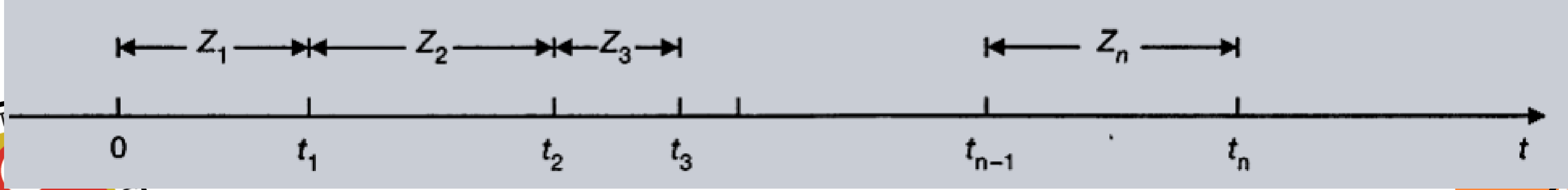
  $X(t)$ is said to have stationary independent increments.

Arrival Process

Let t represent a time variable

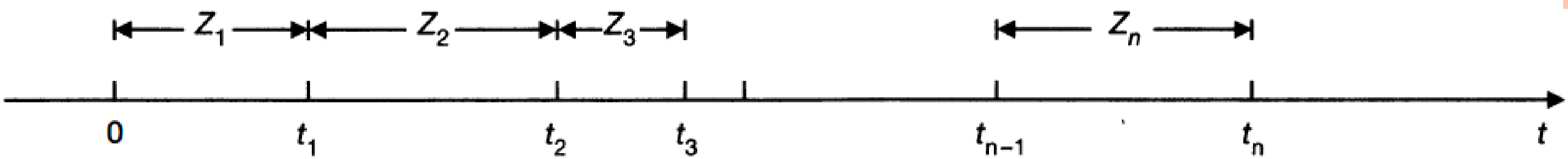
- Suppose an experiment begins at $t = 0$
- Events of a particular kind occur randomly,

  the first at T_1, the second at T_2, and so on.
- The RV (T_i) denotes the time at which the i^{th} event occurs, and
- The values t_i of T_i(i= 1, 2, . . .) are called points of occurrence



Let $Z_n = t_n - t_{n-1}$

Arrival Process



Let $Z_n = T_n - T_{n-1}$ & $T_0 = 0$

Then Z_n denotes the time between the (n - 1)st and the nth events

The sequence of ordered RV's $\{Z_n, \text{n} \geq 1\}$ is sometimes called an interarrival process.

Observe that $T_n = Z_0 + Z_1 + Z_2 + \cdots + Z_n$

The sequence $\{T_n, \text{n} \geq 1\}$ is called Arrival Process

Counting Process

A random process $\{X(t), t \geq 0\}$ is said to be a counting process if:

- X(t) represents the total number of events that have occurred in the interval (0, t)
- $X(t) \geq 0$ and $X(0) = 0$
- $X(t)$ is integer valued & $X(s) \leq X(t)$ if $s < t$
- $X(t)$-$X(s)$ equals to the no of events that have occurred in the interval (s, t)

Counting Process

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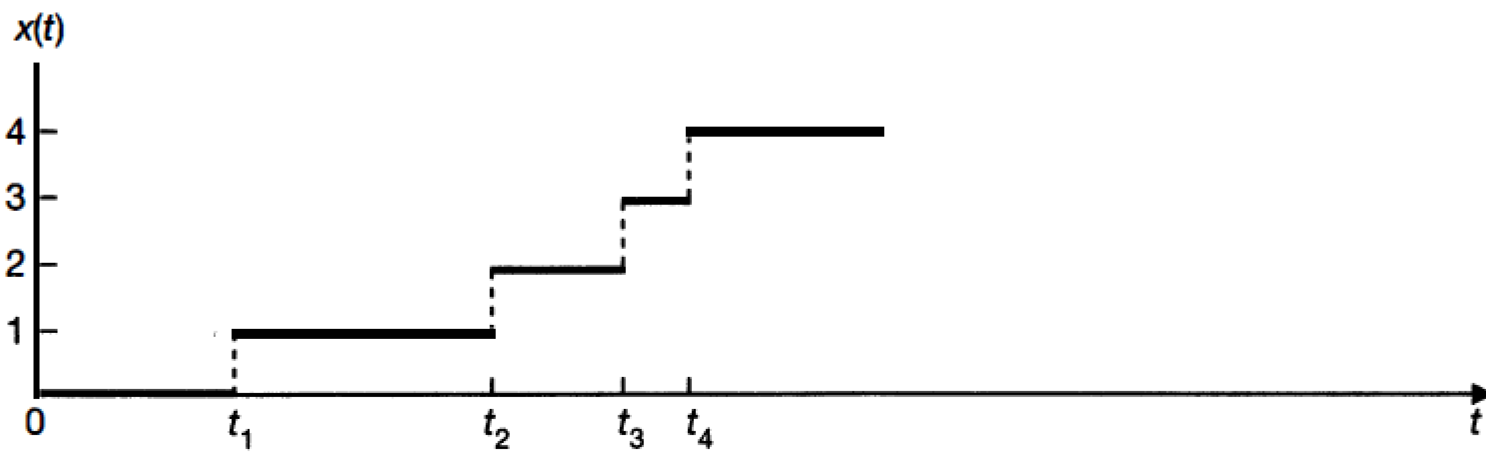
- $X(t)$-$X(s)$ equals to the no of events that have occurred in the interval (s, t)
- $X(t)$ is a independent increment process if the *no of events* which occur in disjoint time intervals are independent
- A counting process $X(t)$ possesses stationary increments if $X(t+h) - X(s+h)$ has the same dist. as $X(t)$- $X(s)$



Poisson counting Process

A counting process X(t) is said to be a Poisson process with rate (or intensity) λ (> 0) if

1. X(O) = 0.
2. X(t) has independent increments.
3. The number of events in any interval of length t is Poisson distributed with mean λt

$$P[X(t+s)-X(s)=n]=e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad n=0,1,2,\ldots \qquad \text{for all} \quad s,t>0,$$

$$E[X(t)] = \lambda t$$

$$\text{Var}[X(t)] = \lambda t$$

Thus, the expected number of events in the unit interval (0, 1), or any other interval of unit length, is just λ (hence the name of the rate or intensity).

Example

Suppose vehicle are passing a bridge with the rate of two per minute.

Q1. In 5 minutes, what is the average number of vehicles?
Q2. what is the variance in 5 minutes ?
Q3. What is the probability of at least one vehicle passing the bridge in that 5 minutes?

To determine the above, the Poisson process is assumed ,where $v(t)$is the number of vehicles in the interval $[0, t]$, with a rate of $\lambda = 2$.

$$P\{V(t) = n\} = \frac{(\lambda t)^n}{n!} e^{-\lambda t}, n = 1, 2, 3, \ldots\ldots$$

For t=5 $$P\{V(t) = 5\} = \frac{5*2^{\ n}}{n!} e^{-5*2} \Rightarrow \mu_v(5) = 10 = \sigma_v^2(5)$$

Q3: Do it yourself : Hint $P\{V(5) \geq 1\}$

Interarrival times for Poisson counting Process

A counting process X(t) is said to be a Poisson process with rate λ (> 0) if

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$$P[X(t+s)-X(s)=n]=e^{-\lambda t}\frac{(\lambda t)^n}{n!} \qquad n=0,1,2,\ldots \qquad \text{for all } s,t>0,$$

The time intervals between successive events (ti) or interarrival times in a Poisson's process X(t) with rate λ are IID exponential with parameter λ

Let Z1, Z2, • • • be the r.v. 's representing the lengths of interarrival times in the Poisson process X(t)

{Z1 > t} takes place if and only if no event of the Poisson process occurs in the interval (0, t),

$$\text{P}\{Z_1 \leq t\} = 1 - P\{Z_1 > t\} = 1 - e^{-\lambda t}$$

Interarrival times for Poisson counting Process

The time intervals between successive events (ti) or interarrival times in a Poisson's process X(t) with rate λ are IID exponential with parameter λ

Let $Z_1, Z_2, \cdots$ be the r.v. 's representing the lengths of interarrival times in the Poisson process X(t).

$\{Z_1 > t\}$ takes place if and only if **no event** of the Poisson process occurs in the interval (0, t),

$$P\{Z_1 \le t\} = 1 - P\{Z_1 > t\} = 1 - e^{-\lambda t}$$

$$P\{Z_2 > t\} = \int P\{Z_2 > t | Z_1 = \tau\} f_1(\tau) d\tau$$
$$= \int P\{X(t+\tau) - X(\tau) = 0] f_1(\tau) d\tau = e^{-\lambda t}$$

which indicates that Z_2 is also an exponential with parameter λ and is independent of Z_1.

Repeating the same argument, we conclude that Z_1 , Z_2, ... are iid exponential r.v.'s with parameter λ

Arrival times for Poisson counting Process

Let $T_n = Z_0 + Z_1 + Z_2 + \cdots + Z_n$ denote the time of the n^{th} event of a Poisson process X(t) with rate λ.

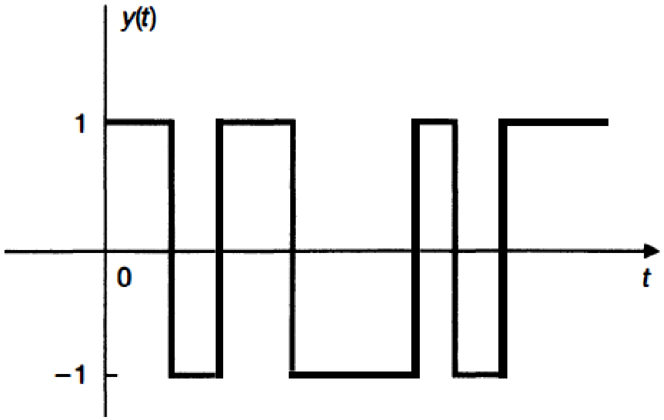
T_n is a gamma r.v. with parameters (n, λ)

we know that Z_n are iid exponential r.v.'s with parameter λ.

It can be proved that the sum of n iid exponential r.v.'s with parameter λ is a gamma RV with parameters (n, λ)

$$f_{T_n}(t) = \begin{cases} \lambda e^{-\lambda t} \dfrac{(\lambda t)^{n-1}}{(n-1)!} & t > 0 \\ 0 & t < 0 \end{cases}$$

Example: Semirandom telegraph signal



Consider the random process

$$Y(t) = (-1)^{X(t)}$$

where $X(t)$ is a Poisson process with rate λ. Thus, $Y(t)$ starts at $Y(0) = 1$ and switches back and forth from $+1$ to -1 at random Poisson times T_i, as shown in Fig. The process $Y(t)$ is known as the *semirandom telegraph signal* because its initial value $Y(0) = 1$ is not random.

Determine the mean and covariance of Y(t)

Example: Semirandom telegraph signal

$$Y(t) = \begin{cases} 1 & \text{if } X(t) \text{ is even} \\ -1 & \text{if } X(t) \text{ is odd} \end{cases}$$

$$P[Y(t)=1] = P[X(t) = \text{even integer}]$$
$$= e^{-\lambda t}\left[1 + \frac{(\lambda t)^2}{2!} + \cdots\right] = e^{-\lambda t}\cosh \lambda t$$

$$P[Y(t)=-1] = P[X(t) = \text{odd integer}]$$
$$= e^{-\lambda t}\left[\lambda t + \frac{(\lambda t)^3}{3!} + \cdots\right] = e^{-\lambda t}\sinh \lambda t$$

$$\mu_Y(t) = E[Y(t)] = (1)P[Y(t) = 1] + (-1)P[Y(t) = -1]$$
$$= e^{-\lambda t}(\cosh \lambda t - \sinh \lambda t) = e^{-2\lambda t}$$

Example: Semirandom telegraph signal

$$Y(t) = \begin{cases} 1 & \text{if } X(t) \text{ is even} \\ -1 & \text{if } X(t) \text{ is odd} \end{cases}$$

Similarly, since $Y(t)Y(t+\tau) = 1$ if there are an even number of events in $(t, t+\tau)$ for $\tau > 0$ and $Y(t)Y(t+\tau) = -1$ if there are an odd number of events, then for $t > 0$ and $t + \tau > 0$,

$$R_Y(t, t+\tau) = E[Y(t)]Y(t+\tau)]$$

$$= (1)\sum_{n \text{ even}} e^{-\lambda\tau} \frac{(-\lambda\tau)^n}{n!} + (-1)\sum_{n \text{ odd}} e^{-\lambda\tau} \frac{(\lambda\tau)^n}{n!}$$

$$= e^{-\lambda\tau} \sum_{n=0}^{\infty} \frac{(-\lambda\tau)^n}{n!} = e^{-\lambda\tau} e^{-\lambda\tau} = e^{-2\lambda\tau}$$

Bernoulli Process

- Let X_1, X_2 …… be independent Bernoulli RVs with P(X_n = 1) = p and P(X_n = 0) = q = 1 – p for all n.

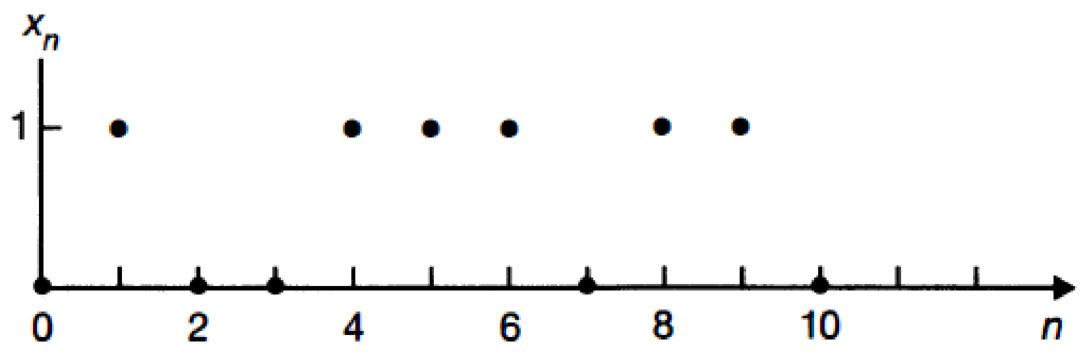
- The collection of RVs $\{X(n)$, n≥ 1 } is a random process, and it is called a Bernoulli process.

- A sample sequence of the Bernoulli process can be obtained by tossing a coin consecutively
  - If a head appears, we assign 1,
  - If a tail appears, we assign 0.

Bernoulli Process

n	1	2	3	4	5	6	7	8	9	10
Coin tossing	H	T	T	H	H	H	T	H	H	T
x_n	1	0	0	1	1	1	0	1	1	0

The sample sequence $\{x_n\}$ obtained above is plotted in Fig.



Random Walk

- Let Z_1, Z_2 …… be independent Bernoulli RVs with P(Z_n = 1) = p and P(Z_n = -1) = q = 1 – p for all n.

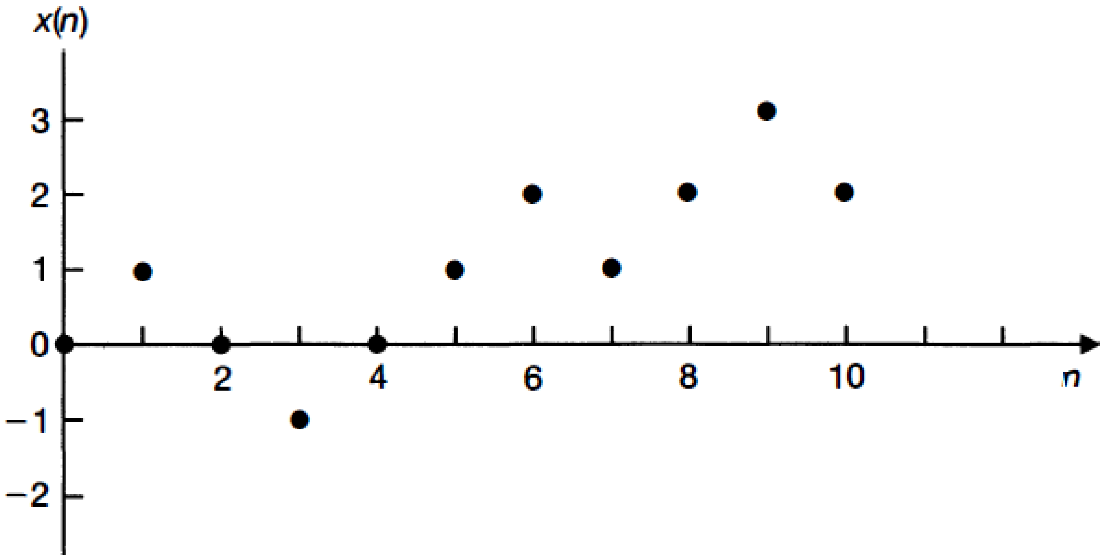
$$X_n = \sum_{i=1}^{n} Z_i \qquad n = 1, 2, \ldots \qquad \text{and } X_0 = 0$$

The collection of RVs $\{X(n)$, n≥ 1 } is a random process, and it is called Random Walk

Random Walk

Repeat the same coin tossing exercise as Bernoulli process

n	0	1	2	3	4	5	6	7	8	9	10
Coin tossing		H	T	T	H	H	H	T	H	H	T
$x(n)$	0	1	0	−1	0	1	2	1	2	3	2



Homework: Find the mean, variance and the autocorrelation of a simple random walk

Wiener Process

A random process $\{X(t), t > 0\}$ is called a Wiener process if

1. X(t) has stationary independent increments.
2. The increment $X(t)$- $X(s)$ $(t > s)$ is normally distributed.
3. $E[X(t)] = 0$
4. $X(0) = 0$

The Wiener process is also known as the Brownian motion process, since it originates as a model for Brownian motion, the motion of particles suspended in a fluid.

Wiener Process

A Wiener process X(t) has stationary independent increments in which the increment $X(t)$- $X(s)$ ($t > s$) is normally distributed with:

1. $\mathrm{E}[X(t)] = 0$
2. VAR $[X(t)] = \sigma^2 t$
3. When σ^2=1, X(t) is called a STANDARD Wiener process

The autocorrelation function of Wiener process $R_x(t, s) = \sigma^2 \min(t, s)$

Wiener Process with Drift

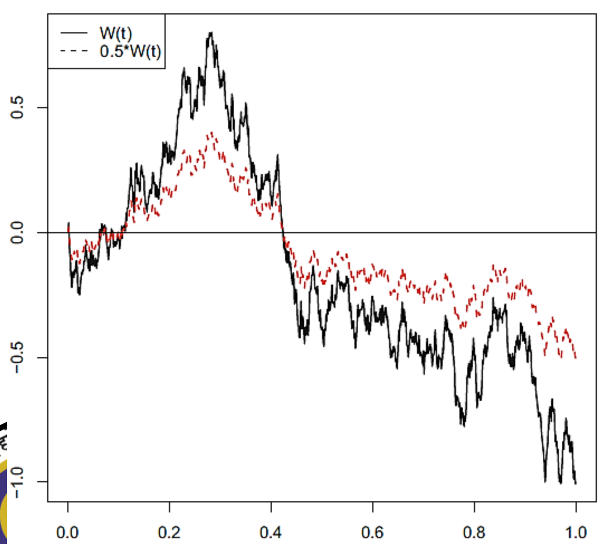
1. X(t) has stationary independent increments.
2. The increment $X(t)$- $X(s)$ ($t > s$) is normally distributed.
3. $\mathrm{E}[X(t)] = \mu t$
4. $X(0) = 0$

The pdf of a standard Wiener process with drift coefficient μ is given by :

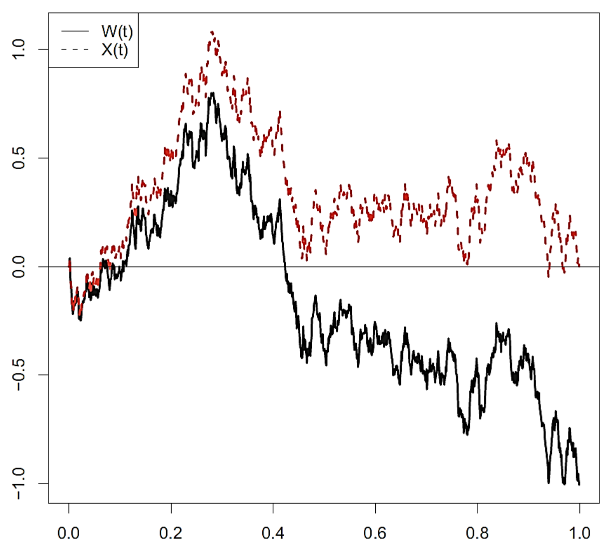
$$\mathrm{f}_{X(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-(x-\mu t)^2/2t}$$

Other processes related to Wiener

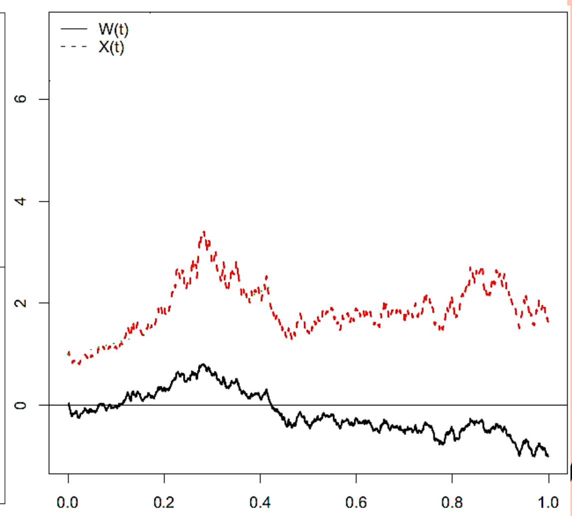
- Brownian Motion: $B(t) = 0.5W(t)$
- Brownian Bridge: $\boldsymbol{X(t)} = B(t) - \boldsymbol{t}B(\boldsymbol{1})$
- Geometric Brownian Motion : $G(t) = e^{\mu t + \sigma W(t)}$



WP and Brownian motion with $\sigma = 0.5$



WP and Brownian bridge ($\sigma = 1$)



Geometric Brownian motion with $\mu = 1.5$ and $\sigma = 1$ along with expectation

Convergence of Random Process

Definitions:

Sequence of RVs: $\{X_n, \mathrm{n} \geq 1\}$; $n \in N$

$E\,[X_n^2] < \infty$

Mean squared Error: MSE $(X_n, X) \coloneqq E\,[(X_n - X)^2]$

Limit in mean square: $\underset{n\to\infty}{\mathrm{l.i.m.}}\; X_n = X$

Is the same way of stating $\lim_{n\to\infty}$ MSE $(X_n, X) \to 0$

Convergence of Random Process

Let $\{X_n\}$ *converge in mean square to* X*. Then it holds for* $n \to \infty$

(a) $E(X_n) \to E(X);$

(b) $E(X_n^2) \to E(X^2)$;

(c) *if* $\{X_n\}$ *is Gaussian, then* X *follows a Gaussian distribution as well*