CE 607: Random Vibration

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Introduction to Stochastic Differential Equations

Stochastic Differential Equations: Intro

- At first, we have an ordinary differential equation (ODE):

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t).$$

- Then we add white noise to the right hand side:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) + \mathbf{w}(t).$$

- Generalize a bit by adding a multiplier matrix on the right:

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t)\,\mathbf{w}(t).$$

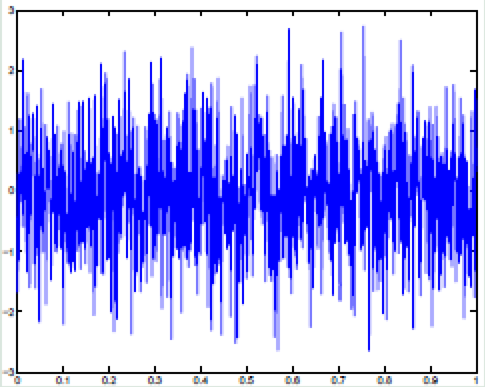
- Now we have a stochastic differential equation (SDE).
- $\mathbf{f}(\mathbf{x}, t)$ is the drift function and $\mathbf{L}(\mathbf{x}, t)$ is the dispersion matrix.

Stochastic Differential Equations: Intro

White noise

1. $\mathbf{w}(t_1)$ and $\mathbf{w}(t_2)$ are independent if $t_1 \neq t_2$.
2. $t \mapsto \mathbf{w}(t)$ is a Gaussian process with the mean and covariance:

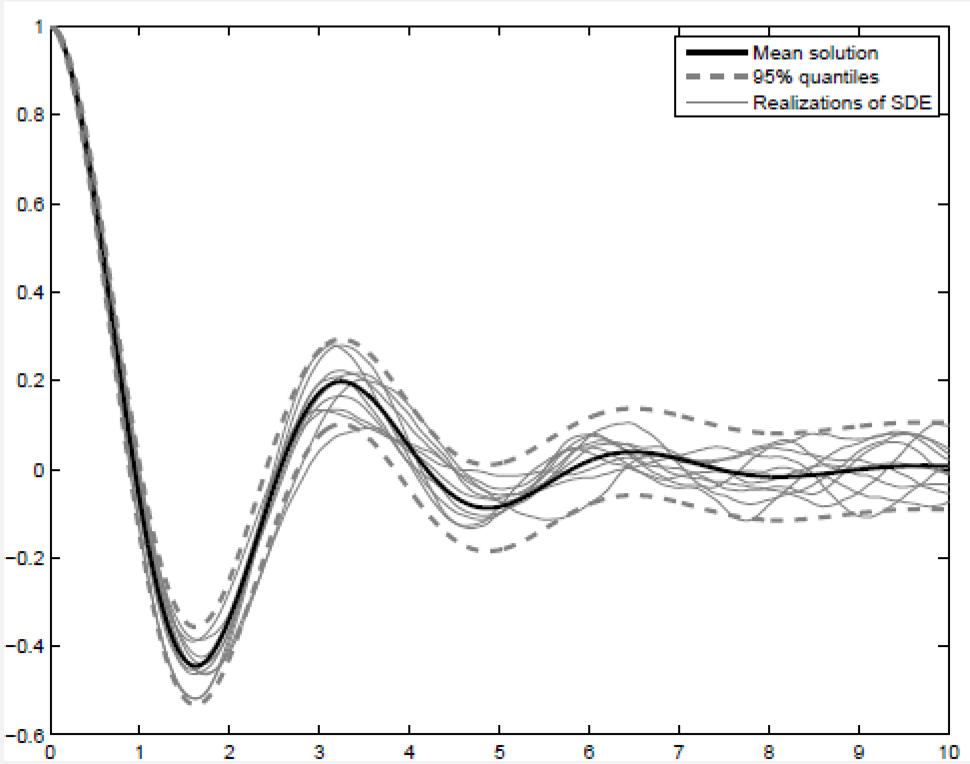
$$\mathrm{E}[\mathbf{w}(t)] = \mathbf{0}$$
$$\mathrm{E}[\mathbf{w}(t)\,\mathbf{w}^{\mathsf{T}}(s)] = \delta(t - s)\,\mathbf{Q}.$$



- $\mathbf{Q}$ is the spectral density of the process.
- The sample path $t \mapsto \mathbf{w}(t)$ is discontinuous almost everywhere.
- White noise is unbounded and it takes arbitrarily large positive and negative values at any finite interval.

Stochastic Differential Equations: Intro

What does a solution of SDE look like?



Paths of stochastic spring model

$$\frac{\mathrm{d}^2 x(t)}{\mathrm{d}t^2} + \gamma \frac{\mathrm{d}x(t)}{\mathrm{d}t} + \nu^2 x(t) = w(t).$$

Attempts to Solution

- Linear time-invariant stochastic differential equation (LTI SDE):

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{F}\,\mathbf{x}(t) + \mathbf{L}\,\mathbf{w}(t), \qquad \mathbf{x}(t_0) \sim \mathrm{N}(\mathbf{m}_0, \mathbf{P}_0).$$

- We can now take a "leap of faith" and solve this as if it was a deterministic ODE:

1. Move $\mathbf{F}\,\mathbf{x}(t)$ to left and multiply by integrating factor $\exp(-\mathbf{F}\,t)$:

$$\exp(-\mathbf{F}\,t)\,\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} - \exp(-\mathbf{F}\,t)\,\mathbf{F}\,\mathbf{x}(t) = \exp(-\mathbf{F}\,t)\,\mathbf{L}\,\mathbf{w}(t).$$

2. Rewrite this as

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\exp(-\mathbf{F}\,t)\,\mathbf{x}(t)\right] = \exp(-\mathbf{F}\,t)\,\mathbf{L}\,\mathbf{w}(t).$$

3. Integrate from t_0 to t:

$$\exp(-\mathbf{F}\,t)\,\mathbf{x}(t) - \exp(-\mathbf{F}\,t_0)\,\mathbf{x}(t_0) = \int_{t_0}^{t} \exp(-\mathbf{F}\,\tau)\,\mathbf{L}\,\mathbf{w}(\tau)\,\mathrm{d}\tau.$$

Attempts to Solution

- Rearranging then gives the solution:

$$\mathbf{x}(t) = \exp(\mathbf{F}\,(t - t_0))\,\mathbf{x}(t_0) + \int_{t_0}^{t} \exp(\mathbf{F}\,(t - \tau))\,\mathbf{L}\,\mathbf{w}(\tau)\;\mathrm{d}\tau.$$

- We have assumed that $\mathbf{w}(t)$ is an ordinary function, which it is not.
- Here we are lucky, because for linear SDEs we get the right solution, but generally not.
- The source of the problem is the integral of a non-integrable function on the right hand side.

Numerical approaches

- We could now attempt to analyze non-linear SDEs of the form

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t)\,\mathbf{w}(t)$$

- We cannot solve the deterministic case—no possibility for a "leap of faith".
- We don't know how to derive the mean and covariance equations.
- What we can do is to simulate by using Euler–Maruyama:

$$\hat{\mathbf{x}}(t_{k+1}) = \hat{\mathbf{x}}(t_k) + \mathbf{f}(\hat{\mathbf{x}}(t_k), t_k)\,\Delta t + \mathbf{L}(\hat{\mathbf{x}}(t_k), t_k)\,\Delta\beta_k,$$

  where $\Delta\beta_k$ is a Gaussian random variable with distribution $\mathrm{N}(\mathbf{0}, \mathbf{Q}\,\Delta t)$.
- Note that the variance is proportional to Δt, not the standard derivation.

Numerical approaches

Equivalent integral equation

- Integrating the differential equation from t_0 to t gives:

$$\mathbf{x}(t) - \mathbf{x}(t_0) = \int_{t_0}^{t} \mathbf{f}(\mathbf{x}(t), t)\, \mathrm{d}t + \int_{t_0}^{t} \mathbf{L}(\mathbf{x}(t), t)\, \mathbf{w}(t)\, \mathrm{d}t.$$

- The first integral is just a normal Riemann/Lebesgue integral.
- The second integral is the problematic one due to the white noise.
- This integral cannot be defined as Riemann, Stieltjes or Lebesgue integral

Numerical approaches

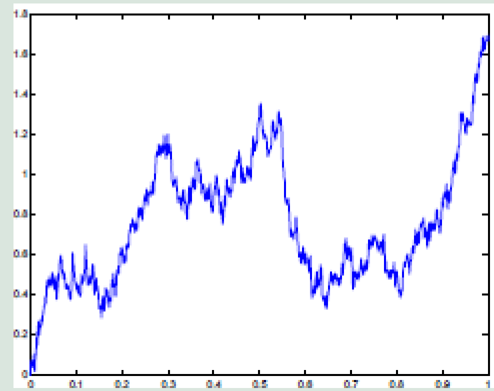
Brownian motion

1. Gaussian increments:

$$\Delta\beta_k \sim \mathrm{N}(0, \mathbf{Q}\,\Delta t_k),$$

   where $\Delta\beta_k = \beta(t_{k+1}) - \beta(t_k)$ and $\Delta t_k = t_{k+1} - t_k$.

2. Non-overlapping increments are independent.



- $\mathbf{Q}$ is the diffusion matrix of the Brownian motion.
- Brownian motion $t \mapsto \beta(t)$ has discontinuous derivative everywhere.
- White noise can be considered as the formal derivative of Brownian motion $\mathbf{w}(t) = \mathrm{d}\beta(t)/\mathrm{d}t$.

Numerical approaches

Itô stochastic differential equations

- Consider the white noise driven ODE

$$\frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t)\,\mathbf{w}(t).$$

- This is actually defined as the Itô integral equation

$$\mathbf{x}(t) - \mathbf{x}(t_0) = \int_{t_0}^{t} \mathbf{f}(\mathbf{x}(t), t)\, dt + \int_{t_0}^{t} \mathbf{L}(\mathbf{x}(t), t)\, d\beta(t),$$

which should be true for arbitrary t_0 and t.

- Settings the limits to t and $t + dt$, where dt is "small", we get

$$d\mathbf{x} = \mathbf{f}(\mathbf{x}, t)\, dt + \mathbf{L}(\mathbf{x}, t)\, d\beta.$$

- This is the canonical form of an Itô SDE.

Numerical approaches

Connection with white noise driven ODEs

- Let's formally divide by $\mathrm{d}t$, which gives

$$\frac{\mathrm{d}\mathbf{x}}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}, t) + \mathbf{L}(\mathbf{x}, t)\,\frac{\mathrm{d}\beta}{\mathrm{d}t}.$$

- Thus we can interpret $\mathrm{d}\beta/\mathrm{d}t$ as white noise **w**.
- Note that we cannot define more general equations

$$\frac{\mathrm{d}\mathbf{x}(t)}{\mathrm{d}t} = \mathbf{f}(\mathbf{x}(t), \mathbf{w}(t), t),$$

because we cannot re-interpret this as an Itô integral equation.

- White noise should not be thought as an entity as such, but it only exists as the formal derivative of Brownian motion.

Numerical approaches

Taylor series of ODEs vs. Itô-Taylor series of SDEs

- Taylor series expansions (in time direction) are classical methods for approximating solutions of deterministic ordinary differential equations (ODEs).
- Largely superseded by Runge–Kutta type of derivative free methods (whose theory is based on Taylor series).
- Itô-Taylor series can be used for approximating solutions of SDEs—direct generalization of Taylor series for ODEs.
- Stochastic Runge–Kutta methods are not as easy to use as their deterministic counterparts
- It is easier to understand Itô-Taylor series by understanding Taylor series (for ODEs) first.

Evolution of SDEs

Newton and Leibniz
Differential Calculus

Louis Bachelier (1900)
Brownian motion

Albert Einstein (1906)
Brownian model

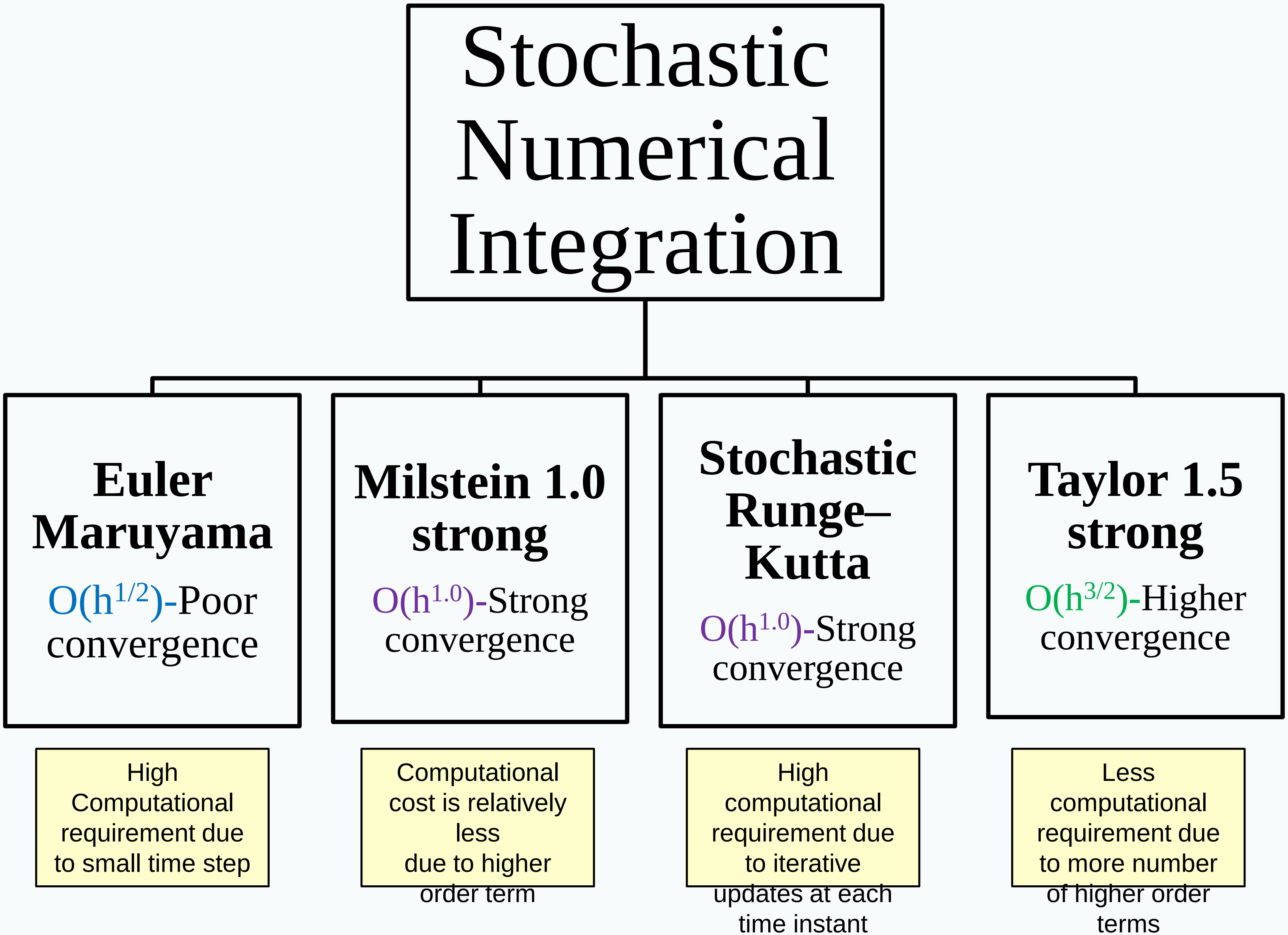
Norbert Wiener (1923)
Wiener process or generalized Brownian motion

Wagner and Platen (1982)
Ito-Taylor expansion

Milstein (1974)
Milstein 1.0 strong method

Gisiro Maruyama (1955)
Euler-Maruyama

Kiyoshi Itô (1944)
Ito calculus



Stochastic Numerical Integration
Euler Maruyama
O(h^1/2)-Poor convergence
Milstein 1.0 strong
O(h^1.0)-Strong convergence
Stochastic Runge–Kutta
O(h^1.0)-Strong convergence
Taylor 1.5 strong
O(h^3/2)-Higher convergence
High Computational requirement due to small time step
Computational cost is relatively less due to higher order term
High computational requirement due to iterative updates at each time instant
Less computational requirement due to more number of higher order terms

Evolution of the present techniques:

Euler-Maruyama Scheme

Milstein 1.0 strong Method

$$x(t_{i+1}) = x(t_i) + a(Y_i)\Delta t + b(Y_i)\Delta B_i + \frac{1}{2} b(Y_i) b'(Y_i)\{(\Delta B_i)^2 - \Delta t\} +$$

$$\{a(Y_i)a'(Y_i) + b^2(Y_i)a''(Y_i)\}\frac{(\Delta t)^2}{2} +$$

$$b(Y_i)a'(Y_i)\Delta Z +$$

$$\{a(Y_i)b'(Y_i) + \frac{1}{2} b^2(Y_i) b''(Y_i)\} (\Delta W \Delta t - \Delta Z)$$

$$+ \frac{1}{2} b(Y_i)\{b(Y_i)b''(Y_i) + b^2(Y_i)\}\left(\frac{\Delta W^2}{3} - \Delta t\right)(\Delta W)$$

Through implicitness in Y_i as:
$Y_i = b\left(Y_i + b(Y_i)\sqrt{\Delta t_i}\right) - b(Y_i)$
Results,
Stochastic Runge-Kutta

Taylor 1.5 strong scheme

STOCHASTIC DEQ: Preliminaries

ITO-Taylor expansion: Preliminaries

Consider the ODE $\dfrac{dx}{dt} = a[x(t)];$ $\quad x(t_0) = x_0 \;\; \& \;\; 0 \leq t_0 \leq T$

The solution can be written as:

$$x(t) = x(t_0) + \int_{t_o}^{t} a[x(s)]ds$$

$$= x_0 + \int_{t_o}^{t} a[x(s)]ds$$

Define: $\dfrac{d\,f[x(t)]}{dt} = \mathcal{L}f[x(t)]$

For a function $f[x(t)]$ we can write

$$\frac{d\,f[x(t)]}{dt} = \frac{\partial\,f[x(t)]}{\partial x}\frac{dx}{dt}$$

Eq. 1

$$\frac{d\,f[x(t)]}{dt} = a[x(t)]\frac{\partial\,f[x(t)]}{\partial x}$$

STOCHASTIC DEQ: Preliminaries

Integral equation : By defining a linear operator $\mathcal{L} = a[x(t)]\,\frac{\partial}{\partial x}$

Eq. 1 can be written in terms of the integral equation as:

$$f[x(t)] = f[x(t_0)] + \int_{t_o}^{t} \mathcal{L} f[x(s)]ds$$

Case-1)

$$f[x(t)] = x(t)$$

$$\frac{dx(t)}{dt} = \mathcal{L}x(t) = a[x(t)]\frac{\partial x}{\partial x} = a[x(t)]$$

$$x(t) = x_0 + \int_{t_o}^{t} a[x(s)]ds \qquad \text{Eq. 2}$$

Case-2)

$$f[x(t)] = a[x(t)]$$

$$\frac{d\ a[x(t)]}{dt} = \mathcal{L}\ a[x(t)] = a[x(t)]\ \frac{\partial}{\partial x} a[x(t)]$$

$$a[x(t)] = a[x(t_0)] + \int_{t_o}^{t} \mathcal{L} a[x(s)]ds$$

Eq. 3

STOCHASTIC DEQ: Preliminaries

From 2 and 3 : $\mathrm{x}(t) = x_0 + \int_{t_o}^{t} \mathcal{L}a(x_0) + \int_{t_o}^{t} \mathcal{L}a[x(s_2)ds_2]ds$

Eq. 4

$$= x_0 + a[x_0]\int_{t_o}^{t} ds_1 + \int_{t_o}^{t}\int_{t_o}^{s_1} \mathcal{L}a[x(s_2)]ds_2\, ds_1$$

Now for, $F = \mathcal{L}\, a[x(t)]$

$$\frac{d\,\mathcal{L}\,a[x(t)]}{dt} = \mathcal{L}\{\mathcal{L}a[x(t)]\} = \mathcal{L}^2 a[x(t)]$$

$$\mathcal{L}\,a[x(t)] = \mathcal{L}a(x_0) + \int_{t_o}^{t} \mathcal{L}^2 a[x(s)]ds \Longleftrightarrow \quad a[x(t)] = a[x(t_0)] + \int_{t_o}^{t} \mathcal{L}a[x(s)]ds$$

$$\mathcal{L}\,a[x(s_2)] = \mathcal{L}a(x_0) + \int_{t_o}^{s_2} \mathcal{L}^2 a[x(s_3)]ds_3$$

Eq. 5

STOCHASTIC DEQ: Preliminaries

From Eq. 4 and 5 :

$$x(t) = x_0 + a(x_0)\int_{t_o}^{t} ds_1 + \int_{t_o}^{t}\int_{t_o}^{s_1}\left\{\mathcal{L}a(x_0) + \int_{t_o}^{s_1}\mathcal{L}^2 a[x(s_3)]ds_3\right\}ds_2\, ds_1$$

$$= x_0 + a(x_0)\int_{t_o}^{t} ds_1 + \mathcal{L}a(x_0)\int_{t_o}^{t}\int_{t_o}^{s_1} ds_2\, ds_1 + \mathcal{L}^2 a[x(s_3)]\int_{t_o}^{t}\int_{t_o}^{s_1}\int_{t_0}^{s_2} ds_3 ds_2 ds_1$$

$$= x_0 + a(x_0)\int_{t_o}^{t} ds_1 + \mathcal{L}a(x_0)\int_{t_o}^{t}\int_{t_o}^{s_1} ds_2\, ds_1 + R$$

$$R = \int_{t_o}^{t}\int_{t_o}^{s_1}\int_{t_0}^{s_2}\mathcal{L}^2 a[x(s_3)]ds_3 ds_2 ds_1$$

2. STOCH-DEQ: ITO-Taylor expansion

Starting point: Diffusion Equation and Ito's lemma

$$dx(t) = a[x(t)]dt + b[x(t)]dB(t), \qquad x(t_0) = x_0$$

Ito's Lemma: $df[x(t)] = f'[x(t)]dx(t) + \frac{1}{2}f''[x(t)]b^2[x(t)]dt$

$$df[x(t)] = f'[x(t)]\{a[x(t)]dt + b[x(t)]dB(t)\} + \frac{1}{2}f''[x(t)]b^2[x(t)]dt$$

$$= \left\{a[x(t)]\frac{\partial f'[x(t)]}{\partial x} + \frac{1}{2}b^2[x(t)]\frac{\partial^2 f[x(t)]}{\partial x^2}\right\}dt + b[x(t)]\frac{\partial f[x(t)]}{\partial x}\,dB(t)$$

$$f[x(t)]$$
$$= f[x(t_0)] + \int_{t_o}^{t}\left\{a[x(s)]\frac{\partial f[x(s)]}{\partial x} + \frac{1}{2}b^2[x(s)]\frac{\partial^2 f[x(s)]}{\partial x^2}\right\}ds + \int_{t_0}^{t} b[x(s)]\frac{\partial f[x(s)]}{\partial x}\,dB(s)$$

Eq. 2.1

STOCH-DEQ: ITO-Taylor expansion

Define: $\mathcal{L}^0 = a[x(s)]\dfrac{\partial}{\partial x} + \dfrac{1}{2}b^2[x(s)]\dfrac{\partial^2}{\partial x^2}$ and $\mathcal{L}^1 = b[x(s)]\dfrac{\partial}{\partial x}$

Eqn. 2.1 becomes $$f[x(t)] = f[x(t_0)] + \int_{t_0}^{t}\mathcal{L}^0 f[x(s)]ds + \int_{t_0}^{t}\mathcal{L}^1 f[x(s)]dB(s)$$ Eq. 2.2

$$f[x(t)] = x(t)$$

$$\mathcal{L}^0 = \left[a[x(s)]\frac{\partial}{\partial x} + \frac{1}{2}b^2[x(s)]\frac{\partial^2}{\partial x^2}\right]x(t) = a[x(s)]\,;\;\; \mathcal{L}^1 = \left[b[x(s)]\frac{\partial}{\partial x}\right]x(t) = b[x(s)]$$

$$x(t) = x(t_0) + \int_{t_0}^{t}\mathcal{L}^0[x(s)]ds + \int_{t_0}^{t}\mathcal{L}^1[x(s)]dB(s)$$

$$= x(t_0) + \int_{t_0}^{t}a[x(s)]ds + \int_{t_0}^{t}b[x(s)]dB(s)$$ Eq. 2.3

STOCHASTIC DEQ: FORMULATION

Further for $f[x(t)] = a[x(s)]$ and using eq. 2.2

$$a[x(s)] = a[x(t_0)] + \int_{t_0}^{s_1} \mathcal{L}^0\, a[x(s_2)]ds_2 + \int_{t_0}^{s_1} \mathcal{L}^1\, a[x(s_2)]dB(s_2) \quad \text{Eq. 2.4}$$

For $f[x(t)] = b[x(s)]$ and using eq. 2.2

$$b[x(s)] = b[x(t_0)] + \int_{t_0}^{s_1} \mathcal{L}^0\, b[x(s_2)]ds_2 + \int_{t_0}^{s_1} \mathcal{L}^1\, b[x(s_2)]dB(s_2) \quad \text{Eq. 2.5}$$

Inserting the results of Eqs. 2.4 and 2.5 into Eq. 2.3

$$x(t) = x(t_0) + \int_{t_0}^{t} \left\{ a[x(t_0)] + \int_{t_0}^{s_1} \mathcal{L}^0\, a[x(s_2)]ds_2 + \int_{t_0}^{s_1} \mathcal{L}^1\, a[x(s_2)]dB(s_2) \right\} ds_1$$

$$+ \int_{t_0}^{t} \left\{ b[x(t_0)] + \int_{t_0}^{s_1} \mathcal{L}^0\, b[x(s_2)]ds_2 + \int_{t_0}^{s_1} \mathcal{L}^1\, b[x(s_2)]dB(s_2) \right\} dB(s_1)$$

$$= x(t_0) + \int_{t_0}^{t} a[x(t_0)]\, ds_1 + \int_{t_0}^{t} b[x(t_0)]\, dB\,(s_1) + R$$

STOCHASTIC DEQ: FORMULATION

$$R = \left[\int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^0 a[x(s_2)]ds_2 + \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^1 a[x(s_2)]dB(s_2)\right]ds_1$$

$$+\left[\int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^1 b[x(s_2)]ds_2 dB(s_1) + \int_{t_0}^{t}\int_{t_0}^{s_1}\mathcal{L}^1 b[x(s_2)]dB(s_2)\right]dB(s_1)$$

R is referred to as the remainder term. We will learn more about this in the next lecture